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Minimal symplectic atlases for Hermitian symmetric spaces of compact type

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Introduction

Our work fits in the context of symplectic geometry and, in particular, the aim of this work is to compute minimal symplectic atlases for classical Hermitian symmetric spaces of compact type.
A symplectic manifold (M, ω) is a $2n$-dimensional manifold M equipped with a closed and nondegenerate 2-form ω. The basic example of symplectic manifold is R^{2n} equipped with the standard symplectic form $\omega_0 = \sum_j \mathrm{d}x_j \wedge \mathrm{d}y_j$. The first interesting result about symplectic geometry is that for all $p \in M$ there is a symplectic embedding φ of the $2n$-dimensional ball equipped with the standard symplectic form $(B^{2n}(r), \omega_0)$ in (M, ω) such that $\varphi(0) = p$. This result gave rise to the introduction of an important symplectic invariant c_G called Gromov width:

$$c_G(M, \omega) = \sup \left\{ \pi r^2 | \quad \exists\, \varphi : \left(B^{2n}(r), \omega_0\right) \to (M, \omega) \right\}$$

In [21] Rudyak-Schlenk introduced the invariant:

$$S_B(M, \omega) := min\{k | M = \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_k\}$$

where $\mathcal{B}$ is the image of a Darboux chart $\varphi(B^{2n}) \subset M$. This is the minimal number of symplectic charts needed to cover (M, ω). An immediate lower bound for $S_B(M, \omega)$ is $\lambda(M, \omega) := \max\{\Gamma(M, \omega); \mathrm{B}(M)\}$ where $\mathrm{B}(M)$ is the number of charts of a minimal (not necessarily symplectic) atlas and

$$\Gamma(M, \omega) := \left\lfloor \frac{Vol(M, \omega) n!}{c_G(M, \omega)^n} \right\rfloor + 1,$$

the braket $\lfloor x \rfloor$ denoting the maximal integer smaller than or equal to x. Their main result about minimal atlases in [21] is the following

Theorem 1. *i) If* $\lambda(M,\omega) \geq 2n+1$ *then* $S_B(M,\omega) = \lambda(M,\omega)$.

ii) If $\lambda(M,\omega) < 2n+1$ *then* $n+1 \leq \lambda(M,\omega) \leq S_B(M,\omega) \leq 2n+1$.

It is then clear that problem of computing the invariant $S_B(M,\omega)$ is strictly related to the knowledge of the Gromov width of (M,ω).
If we consider a $2n$-dimensional projective variety M in $\mathbb{C}P^d$ the result can be presented in terms of the degree of the embedding $F : M \to \mathbb{C}P^d$. Indeed such a manifold is Kähler when equipped with the restriction ω_M of the Fubini-Study form ω_{FS} of $\mathbb{C}P^d$. Moreover the volume of such a manifold is related to the volume of $\mathbb{C}P^n$ by the formula $\mathrm{Vol}(M) = \deg(F)\mathrm{Vol}(\mathbb{C}P^n)$. Thus we get the following:

Corollary 2. *Let* (M,ω_M) *be a projectively induced Kähler manifold with* $c_G(M,\omega_M) = \pi r^2$*. If* $\dfrac{\deg(F)\pi^n}{c_G(M,\omega_M)^n} \geq 2n$ *then* $S_B(M,\omega_M) = \deg(F)/r^{2n}+1$.

In particular this corollary implies that for a projectively induced Kähler manifold (M,ω_M) it is sufficient to know $c_G(M,\omega_M)$ and the degree of the embedding in order to compute the invariant $S_B(M,\omega_M)$. Unfortunately computing the Gromov width of a symplectic manifold is usually a very delicate problem. However in [12] Loi-Mossa-Zuddas calculated the Gromov width of Hermitian symmetric spaces of compact type. In this thesis, using the above results, we prove the following:

Theorem 3. *Let* (M,ω) *be an irreducible compact Hermitian symmetric spaces of type I,II or III. Then* $S_B(M,\omega_M) = \deg(F)+1$ *when the dimension of* M *is sufficiently large.*

Moreover, using the work of Loi-Mossa-Zuddas, we are able to extend this result to product of these spaces. Unfortunately the irreducible compact domain of type IV Q_n does not satisfy the hypothesis of corollary 2 thus we cannot compute S_B using the same arguments.
Nevertheless, in the last part of the thesis, we provide an explicit construction of a full symplectic embedding of Q_n, namely a collection of symplectic embeddings $\varphi_i : B^{2n}(1) \to Q_n$ such that $\bigcup_i \overline{\varphi_i(B^{2n}(1))} = Q_n$.

Chapter 1

Symplectic geometry

This chapter is dedicated to the basic notions the reader will need throughout the thesis. It is meant to be an overview of some concept of symplectic geometry we will use in next chapters. Helpful introductive readings on symplectic geometry are [17, 18]. We will not present an exhaustive study of symplectic geometry, neither of the other fields we consider along the chapter, indeed we only give an introduction to the topics and fix the notation. However we will introduce some important ideas and results which justify the interest in this subject. The chapter is organized as follows. In the first section we introduce symplectic manifolds and give an overview of the properties which are related to the symplectic structure. In section 2 we focus on the problem of symplectic embedding and present Gromov's nonsqueezing theorem which plays an important role throughout next chapters. Moreover we introduce symplectic capacities and, in particular, Gromov width. In the last section we focus on Hermitian symmetric spaces, which we regard from the point of view of Lie theoretic methods.

1.1 Symplectic manifolds

Let us start with the definition of symplectic vector space:

Definition 1.1. A symplectic vector space (V, ω) is a finite dimensional real vector space V with a bilinear form ω satisfying the following properties

1. $\omega(u, v) = -\omega(v, u)$,

2. $\forall v \neq 0 \in V$ there is $u \in V$ s.t. $\omega(u,v) \neq 0$ (nondegeneracy)

An example that plays an essential role in this context is $(\mathbb{R}^{2n}, \omega_0)$ with

$$\omega_0(u,v) = \langle Ju, v\rangle \qquad \forall\, u,v \in \mathbb{R}^{2n}$$

where $\langle , \rangle$ is the Euclidean product in $\mathbb{R}^{2n}$ and J is the standard complex structure

$$J = \begin{pmatrix} 0 & Id \\ -Id & 0 \end{pmatrix}$$

with respect to the splitting $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$. This splitting allows us to identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ via the map

$$(x,y) \in \mathbb{R}^{2n} \mapsto x + iy \in \mathbb{C}^n.$$

Note that under this identification the linear map J corresponds to the multiplication by $-i$.
Back to the general context we want to point out that the bilinear form ω gives rise to a notion of orthogonality: we say that two vectors $u, v \in V$ are orthogonal to each other if $\omega(u,v) = 0$. If E is a linear subspace of V then we call $E^\perp$ its orthogonal complement. As a direct consequence of nondegeneracy property we get

$$\dim E + \dim E^\perp = \dim V$$

However this notion of orthogonality is quite different from the usual one. For instance the subspaces E and $E^\perp$ do not need to be complementary subspaces: e.g. if $\dim E = 1$ then $E \subset E^\perp$ since for all $v \in V$ it holds

$$\omega(v,v) = -\omega(v,v) = 0.$$

If $E \subset V$ is a subspace such that $E \subset E^\perp$ then E is called an isotropic subspace. Moreover the restriction of ω to a linear subspace E is not necessarily nondegenerate, if it happens we call E a symplectic subspace and we have

$$V = E \oplus E^\perp.$$

The following proposition contains the most important properties of symplectic vector spaces.

Proposition 1.2. *The dimension of a symplectic vector space* (V, ω) *is even. Moreover if* $\dim V = 2n$ *then there exist a basis* $e_1, \dots, e_n, f_1, \dots, f_n$ *of* V *satisfying, for* $i, j = 1, \dots n$

$$\omega(e_i, e_j) = 0$$
$$\omega(f_i, f_j) = 0$$
$$\omega(e_i, f_j) = \delta_{i,j}$$

Such a basis is called a symplectic (or canonical) basis of V*.*

Proof. Choose a non-zero vector $e_1 \in V$. By nondegeneracy of ω there exist $v \in V$ such that $\omega(e_1, v) \neq 0$. Now normalize $f_1 = \alpha v$ so that $\omega(e_1, f_1) = 1$. We see that E =span$\{e_1, f_1\}$ is a 2-dimensional linear subspace of V. If $\dim V = 2$ the proof is complete, otherwise we apply the same argument to $E^\perp$ and we prove the claim in finitely many steps. □

Proposition 1.2 implies that, if $u, v \in V$ with respect to the symplectic basis are given by

$$u = \sum_{i=1}^{n} x_i e_i + x_{n+i} f_i$$
$$v = \sum_{i=1}^{n} y_i e_i + y_{n+i} f_i$$

then

$$\omega(u, v) = \langle Jx, y \rangle \qquad x, y \in \mathbb{R}^{2n}$$

In addition the subspaces V_j =span$\{e_j, f_j\}$ are symplectic and orthogonal to each other so that V can be decomposed as

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_n.$$

Now let us consider linear a map $A : (V, \omega_V) \to (W, \omega_W)$ between symplectic vector spaces such that $A^*\omega_W = \omega_V$, where A^* is the so-called pullback of

A that is $(A^*\omega_W)(u,v) = \omega_W(Au, Av)$. Such a map is called symplectic. The following proposition gives an essential characterization of symplectic vector spaces.

Proposition 1.3. *If* (V_1, ω_1) *and* (V_2, ω_2) *are two symplectic spaces of the same dimension then there exist a linear isomorphism* $A : V_1 \to V_2$ *such that*

$$A^*\omega_2 = \omega_1$$

.

Proof. This comes from Proposition 1.2. We choose symplectic bases (e_j, f_j) of V_1 and $(\hat{e}_j, \hat{f}_j)$ of V_2 and we define $A : V_1 \to V_2$ by

$$Ae_j = \hat{e}_j, \qquad Af_j = \hat{f}_j$$

for $1 \leq j \leq n$. Then by definition of symplectic basis we get $A^*\omega_2 = \omega_1$. □

It means that two symplectic spaces of same dimension are symplectically indistinguishable and makes the study of symplectic vector spaces not really interesting. Thus we want to generalize from vector spaces to manifolds and we will see that new properties arise in this context.

Definition 1.4. A symplectic manifold (M, ω) is a differentiable manifold M equipped with a closed nondegenerate 2-form ω. Here nondegeneracy condition means that for every tangent space T_pM if $\omega_p(u,v) = 0$ for all $v \in T_pM$ then $u = 0$.

From this definition we clearly see that every tangent space T_pM is a symplectic vector space with the bilinear form ω_p and we conclude that M must be even-dimensional. In the context of symplectic manifolds there exist a "standard model" that is $(\mathbb{R}^{2n}, \omega_0)$ where we write a point in $\mathbb{R}^{2n}$ as $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ and the so-called standard symplectic form is

$$\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

Other examples of symplectic manifold are:

1) Any orientable surface Σ equipped with a volume form ν is a symplectic manifold since a volume form is closed and nondegenerate.

2) The complex projective space $\mathbb{C}P^n$ with the so-called Fubini-Study form ω_{FS}, which in homogeneous coordinates $[z_0 : \ldots : z_n]$, is given by

$$\omega_{FS} = \frac{i}{2\pi}\partial\bar{\partial}\log(|z_0|^2 + \cdots + |z_n|^2). \tag{1.1}$$

3) The product of any two symplectic manifolds (M_1, ω_1) and (M_2, ω_2) is symplectic with the form $\omega_1 \oplus \omega_2$.

Definition 1.4 also implies that $\beta = (\wedge\omega)^n$ is a volume form on M due to the nondegeneracy of ω. The volume of (M, ω) will then be

$$\frac{1}{n!}\int_M \beta \tag{1.2}$$

We will see in this chapter that symplectic geometry is substantially dissimilar from Riemannian geometry. The first difference we can observe by now is that every symplectic manifold carries a Riemannian structure while not every $2n$-dimensional manifold admits a symplectic structure. Consider for example the sphere S^{2n} and suppose ω is a symplectic form on S^{2n}. In particular $\omega = d\alpha$ for some 1-form α since $H^2(S^{2n})$ is trivial. That means the volume form $\beta = (\wedge\omega)^n$ is exact that is it can be written as $\beta = d\gamma$ where $\gamma = \omega \wedge \cdots \wedge \omega \wedge \alpha$. In conclusion, by Stokes Theorem, we have

$$\int_{S^{2n}} \beta = \int_{\partial S^{2n}} \gamma = 0$$

which is impossible for a volume form.

The following Theorem justifies the term "standard model" we used introducing $(\mathbb{R}^{2n}, \omega_0)$.

Theorem 1.5 (Darboux)**.** *Let (M, ω) be a symplectic $2n$-dimensional manifold and $p \in M$. There exist coordinates (U, φ) with $U \subset \mathbb{R}^{2n}$ such that $\varphi(0) = p$ and*

$$\varphi^*\omega = \omega_0.$$

Proof. If we choose any local coordinates we can assume that ω is a symplectic form on $\mathbb{R}^{2n}$ depending on $z \in \mathbb{R}^{2n}$ and that p corresponds to the origin. Furthermore by a linear change of coordinates we can manage ω to

$$\omega(0) = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

We can do that thanks to Proposition 1.2. Now the goal is to find a local diffeomorphism ϕ in a neighborhood of $z = 0$ leaving the origin fixed and such that $\phi^*\omega = \omega_0$.
The technique employed to prove this is called the *deformation method* of J. Moser. We define a family of forms interpolating ω and ω_0 by

$$\omega_t = \omega_0 + t(\omega - \omega_0) \qquad 0 \leq t \leq 1$$

and look for a family of diffeomorphisms ϕ_t such that $\phi_0 = Id$ and

$$\phi_t^*\omega_t = \omega_0 \qquad 0 \leq t \leq 1. \tag{1.3}$$

Our solution will be then the diffeomorphism ϕ_1. We want to construct ϕ_t as the flow of a vector field X_t. Thus we look at the conditions that the vector field X_t must satisfy. Differentiating (1.3) we get

$$0 = \frac{d}{dt}(\phi_t)^*\omega_t = (\phi_t)^*\left\{ \pounds_{X_t}\omega_t + \frac{d}{dt}\omega_t \right\}$$

where $\pounds_X$ is the Lie derivative of the vector field X. By Cartan identity and assuming $d\omega_t = 0$ we get

$$0 = \left\{ d(\iota_{X_t}\omega_t) + \omega - \omega_0 \right\}$$

then X_t must satisfy the equation

$$0 = (\iota_{X_t}\omega_t) + \omega - \omega_0 \tag{1.4}$$

In order to solve this equation note that, since $\omega - \omega_0$ is closed, then it is locally exact and there exist a 1-form λ so that

$$\omega - \omega_0 = d\lambda \ \ and \ \ \lambda(0) = 0$$

Observe that since $\omega_t(0) = \omega_0$ the forms ω_t are nondegenerate in an open neighborhood of the origin. Hence there is a unique vector field X_t which solves the equation (1.4) and it is given by

$$\iota_{X_t}(\omega_t) = \omega_t(X_t, \cdot) = -\lambda \qquad 0 \leq t \leq 1$$

There is an open neighborhood of $z = 0$ in which the flow ϕ_t of X_t exist for $0 \leq t \leq 1$, moreover $\phi_t(0) = 0$ and $\phi_0 = Id$ since $\lambda(0) = 0$ implies $X_t(0) = 0$. By construction this family of diffeomorphism satisfies

$$\frac{d}{dt}(\phi_t)^*\omega_t = 0, \quad 0 \leq t \leq 1.$$

Thus $(\phi_t)^*\omega_t = (\phi_0)^*\omega_0 = \omega_0$ for $0 \leq t \leq 1$ and this proves the Theorem. □

Remark. In short Darboux Theorem means that every $2n$-dimensional symplectic manifold looks locally like $\mathbb{R}^{2n}$ with the standard symplectic form that is: there is no local symplectic invariant other than the dimension.

Extending this concept we will call $\phi : (M_1, \omega_1) \to (M_2, \omega_2)$ a symplectomorphism between two symplectic manifold (M_1, ω_1) and (M_2, ω_2) if ϕ is a diffeomorphism such that $\phi^*\omega_2 = \omega_1$. If $\dim(M_1) \leq \dim(M_2)$ we call symplectic an embedding ϕ such that

$$\phi^*\omega_2 = \omega_1.$$

This makes symplectic geometry sharply different from Riemannian geometry where one can easily find local invariants (consider for example the Gaussian curvature). We should then focus on the construction of global invariants.

The first (even though trivial) example is the symplectic volume defined in equation (1.2). In fact if $\phi : (M_1, \omega_1) \to (M_2, \omega_2)$ is a symplectomorphism and $\beta_i = (\wedge\omega_i)^n$ $(i = 1, 2)$ then $\phi^*\beta_2 = \beta_1$. Furthermore since ϕ preserves orientation we get

$$\int_{M_1} \phi^* \beta_2 = \int_{M_2} \beta_2$$

and we conclude that

$$\int_{M_1} \beta_1 = \int_{M_2} \beta_2.$$

Conversely one cannot expect that two symplectic manifolds with same volume must be symplectomorphic. Still we will show that it holds in the special case of closed connected oriented 2-manifolds. Closed connected symplectic surfaces can be indeed classified by the Euler characteristic and total volume. This result comes from the following more general

Theorem 1.6 (Moser)**.** *Let M be a closed connected m-dimensional manifold and let α and β be two volume forms such that*

$$\int_M \alpha = \int_M \beta.$$

Then there is a diffeomorphism ϕ such that $\phi^*\beta = \alpha$.

Proof. The technique is the same we used to prove Darboux Theorem. The only difference is that here we are looking for a global result instead of a local one. It is obtained by compactness of M and by the existence of a $(m-1)$-form γ on M such that $d\gamma = (\alpha - \beta)$. The existence of γ is given by the fact that $(\alpha - \beta)$ is closed and therefore exact since $H^{m+1}(M) = 0$. □

Another concept that arises naturally with the symplectic structure is that of Hamiltonian vector field. Let then (M, ω) be a symplectic manifold. In order to introduce Hamiltonian vector fields note that since ω is non-degenerate it induces an isomorphism between vector fields and 1-forms by $X \mapsto \iota_X \omega$. In particular if $H : M \to \mathbb{R}$ is a smooth function then we can consider the vector field X_H correspondent to the 1-form dH:

$$\iota_{X_H}\omega = \omega(X_H, \cdot) = -dH. \tag{1.5}$$

This distinguished vector field is called the Hamiltonian vector field belonging to the function H. Since $d\omega = 0$ combining equation (1.5) with Cartan identity and $ddH = 0$ we deduce

$$\pounds_{X_H}\omega = 0$$

Now if φ^t is the flow of X_H we see that

$$\frac{d}{dt}(\varphi^t)^*\omega = (\varphi^t)^*\pounds_{X_H}\omega = 0.$$

It follows then from $(\varphi^0)^*\omega = \omega$ that the flow of an Hamiltonian vector field leaves the symplectic form invariant. Hamiltonian vector fields are also invariant under symplectomorphisms as shows the following:

Proposition 1.7. *If $u : M \to M$ is a symplectomorphism then for every smooth function $H : M \to \mathbb{R}$ it satisfies the equation*

$$u^*X_H = X_{H\circ u}$$

Proof. The claim follows from the nondegeneracy of ω and the calculation

$$\begin{aligned}\iota_{X_{H\circ u}}\omega &= -d(H\circ u) = -u^*(dH)\\ &= u^*(\iota_H\omega) = \iota_{u^*X_H}(u^*\omega)\\ &= \iota_{u^*X_H}\omega.\end{aligned}$$

□

In order to conclude our brief introduction to symplectic manifolds we want to underline that given a symplectic structure on a even-dimensional manifold we can construct an auxiliary structure which assume an important role.

Proposition 1.8. *Let (M,ω) be a symplectic manifold. There exist on M an almost complex structure J and a Riemannian metric $\langle\cdot,\cdot\rangle$ such that*

$$\omega(X, JY) = \langle X, Y\rangle \tag{1.6}$$

The condition above is called taming condition and we will call such an almost complex structure an ω-tame J. Note that in $\mathbb{R}^{2n}$ the triple

(ω_0, g_0, J_0), where J_0 is the standard complex structure, satisfies the taming condition. Moreover proposition 1.8 gives an expression of a Hamiltonian vector field X_H in terms of the gradient ∇H of the generating function H with respect to the metric $\langle \cdot, \cdot \rangle$ that is

$$X_H(p) = J\nabla H(p) \in T_pM.$$

If (M, ω) is a complex manifold we can define a richer structure on M:

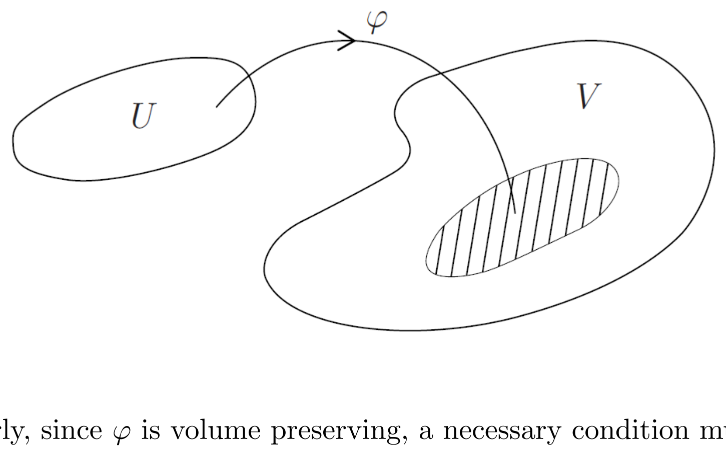
Definition 1.9. A complex manifold M is called a Kähler manifold if it admits a symplectic form ω and a Hermitian metric g such that for all $X, Y \in TM$

$$g(X, JY) = \omega(X, Y)$$

where J is the complex structure on M. We will then call ω a Kähler form and g a Kähler metric.

1.2 Symplectic capacities

We will see in this section that symplectic geometry is much more rigid than it seems at first glance. One of the first problems which arises is that of symplectic embedding. Starting by a simple case one can ask which are the conditions for the existence of a symplectic embedding $\varphi : U \to V$ from an open domain U in $\mathbb{R}^{2n}$ to another open domain V.



Clearly, since φ is volume preserving, a necessary condition must be $\mathrm{Vol}(U) \leq \mathrm{Vol}(V)$ and it turns out that the condition $\mathrm{Vol}(U) < \mathrm{Vol}(V)$ is already sufficient to guarantee the existence of a volume preserving diffeomorphism. Thus the question is whether there are symplectic obstruction to the existence of a symplectic embedding. Consider for example the ball $B^{2n}(R)$ of radius R in $\mathbb{R}^{2n}$ and the cylinder

$$\hat{Z}^{2n}(r) = \{(x, y) \in \mathbb{R}^{2n} |\ x_1^2 + x_2^2 < r^2\}.$$

In this case the volume of $B^{2n}(R)$ is finite unlike the volume of $\hat{Z}^{2n}(r)$, and we can construct a symplectic embedding $\varphi : B^{2n}(R) \to \hat{Z}^{2n}(r)$ for every $r, R \in \mathbb{R}$ which is indeed given by

$$\varphi(x, y) = (\varepsilon x_1, \varepsilon x_2, \ldots, x_n, \frac{1}{\varepsilon} y_1, \frac{1}{\varepsilon} y_2, \ldots, y_n)$$

for ε sufficiently small. The problem changes radically if we replace $\hat{Z}^{2n}(r)$ by

$$Z^{2n}(r) = \{(x, y) \in \mathbb{R}^{2n} |\ x_1^2 + y_1^2 < r^2\}.$$

We can notice that in this case case the plane $\mathrm{Span}\{x_1; y_1\}$ is a symplectic subspace in contrast with the first case. One could try, in analogy with the previous situation, to define the embedding

$$\psi(x, y) = (\varepsilon x_1, \frac{1}{\varepsilon} x_2, \ldots, x_n, \varepsilon y_1, \frac{1}{\varepsilon} y_2, \ldots, y_n)$$

Unfortunately in this case ψ is a volume preserving embedding for ε small enough but it is symplectic only if $\varepsilon = 1$ that is when $R \leq r$.

One can think to do better with nonlinear maps but next theorem shows that it is not possible.

Theorem 1.10 (Gromov's Nonsqueezing Theorem)**.** *There exist a symplectic embedding $\varphi : B^{2n}(R) \to Z^{2n}(r)$ if and only if $R \leq r$.*

Gromov's nonsqueezing theorem was the first step to understand the rigidity of symplectic geometry. It gives us the idea that the behaviour of symplectic embeddings might be very different from how one can imagine it.

In order to present the idea of Gromov's proof we introduce the concept of J-holomorphic curves. Consider the set $\mathcal{J}$ of all ω-tame J that is nonempty

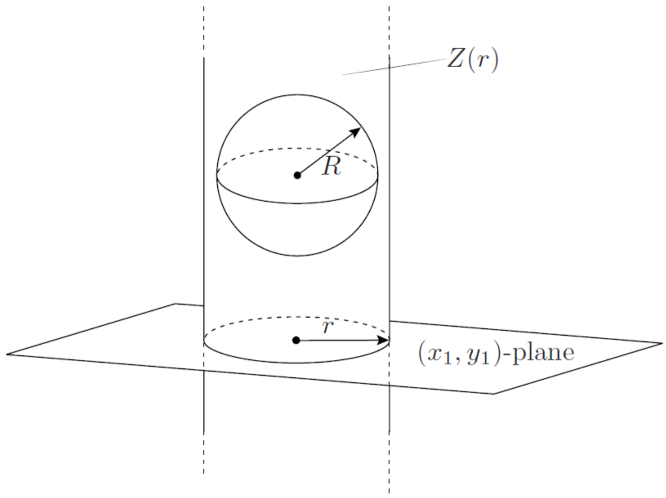


Figure 1.1: Embedding $B^{2n}(R)$ in $Z^{2n}(r)$

by proposition 1.8. It can be proved that $\mathcal{J}$ is contractible, hence one can find invariants of (M, ω) looking at those of the almost complex manifold (M, J) which do not depend on the choice of ω-tame J. In order to find his invariant Gromov looked at the maps of Riemannian surfaces with complex structure j:

$$u : (\Sigma, j) \to (M, \omega)$$

satisfying the generalized Cauchy-Riemann condition

$$du \circ j = J \circ du \tag{1.7}$$

Such maps are called J-holomorphic curves. Since equation (1.7) is elliptic the solution spaces have nice properties.

In particular it turns out that, for a ω-tame J, the space $\mathcal{M}(A, J)$ of solution in a homology class A is finite dimensional. Moreover, even if $\mathcal{M}(A, J)$ is not compact because curves can degenerate, the taming condition allow us to understand and control these degenerations and then compactify the space of solutions.

Now, since the space $\mathcal{J}$ is path-connected, given any two ω-tame J we can construct a path J_t joining them such that so that the spaces of solutions $\mathcal{M}(A, J_t)$ for all $t \in [0, 1]$ give us a cobordism between the solution spaces at 0 and 1. In many cases this cobordism is compact meaning that the properties of $\mathcal{M}(A,J)$ which are cobordism invariant do not depend on the choice of J.

Then Gromov defined his invariant counting the number of J-holomorphic curves (with given genus in a given homology class) that pass through a fixed number of points or cycles.

We will show now how Gromov used this construction to prove theorem 1.10. Note that the symplectic area of a J_0-holomorphic curve S properly embedded in $B^{2n}(R)$ passing through the origin is at least πR^2. This comes from the fact that J_0-holomorphic curves are complex curves in the usual sense and so are minimal surfaces with respect to g_0. Moreover one can easily see that the g_0-area of a complex surface S equals the symplectic area $\int_S \omega_0$. Thus the claim follows from the well known fact that the minimal surface through the origin is the flat disc.

We now consider an embedding of the ball $B^{2n}(R)$ in the cylinder $Z^{2n}(r)$. On the image we have the pushforward of J_0 which we can extend to a ω_0-tame J near the boundary of the cylinder. Then J extends to the compact manifold $S^2 \times \mathbb{R}^{2n-2}$ obtained by closing up the cylinder.

Now the product almost complex structure on $S^2 \times \mathbb{R}^{2n-2}$ is generic (that means we are in the situation we described above). Moreover if J is the product almost complex structure than there is a unique (up to reparametrization) flat J-holomorphic 2-sphere through every point.

Thus the value of the Gromov invariant that counts the number of spheres in the homology class $[S^2 \times pt]$ is 1. Since this value does not depend on the ω-tame J, there is at least one J-holomorphic sphere Σ through the image of the center of the ball where J equals the pushforward of J_0 on the image of the ball.

Note that the symplectic area of Σ depends only on its homology class and hence it is $\pi(r+\varepsilon)^2$ for arbitrary small ε (the term ε appears with the compactification of $Z^{2n}(r)$).

We now look at the inverse image S of Σ in $B^{2n}(R)$:

For what we have seen above the area of S must be at least πR^2, but since symplectomorphisms preserve area it has to be less than the area of Σ

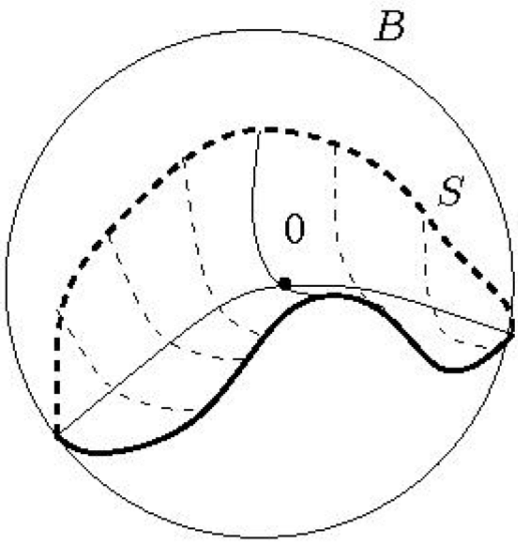


Figure 1.2: The surface S in $B^{2n}(R)$

which is $\pi(r+\varepsilon)^2$. Since ε is arbitrary small we get $R \leq r$ which proves the theorem.

This theorem has been generalized later generalized by McDuff and Lalonde ([10]) in the following form:

Theorem 1.11. *For any symplectic $2n$-dimensional manifold M there exist a symplectic embedding*

$$\varphi : B^{2n+2}(r) \to B^2(R) \times M$$

if and only if $r \leq R$.

Remark. Roughly speaking the symplectic obstruction to the existence of a symplectic embedding $\varphi : (M, \omega_M) \to (N, \omega_N)$ is related to the size of surfaces in the manifolds M and N, that means symplectic geometry regards surfaces rather than curves.

From this concept and Darboux theorem arises the key idea to define a global invariant called Gromov width. Indeed we can symplectically embed a ball $B^{2n}(\varepsilon)$ of radius ε small enough in every symplectic manifold (M, ω)

of dimension $2n$ thus it makes sense to ask which is the bigger ball that can be embedded in M.

Definition 1.12. The Gromov width $c_G(M,\omega)$ of a symplectic $2n$-dimensional manifold is

$$c_G(M,\omega) = \sup\left\{\pi r^2 \middle|\ \exists\, \varphi : \big(B^{2n}(r),\omega_0\big) \to (M,\omega)\right\} \tag{1.8}$$

where φ is a symplectic embedding.

It is then clear from the previous remark on the proof of nonsqueezing theorem why the Gromov width of a manifold is defined as in (1.8). We will now investigate the properties of Gromov width in a more general context. Gromov's nonsqueezing theorem gave rise to the concept of symplectic capacity which is a generalization of that of Gromov width.
The definition we give is due to Ekeland and Hofer ([2]):

Definition 1.13. A **symplectic capacity** is a functor c which assigns to every symplectic manifold (M,ω) a nonnegative (possibly infinite) number $c(M,\omega)$ and satisfies the following properties:

1. (**monotonicity**) If there exist a symplectic embedding $(M_1^{2n},\omega_1) \hookrightarrow (M_2^{2n},\omega_2)$ then $c(M_1^{2n},\omega_1) \le c(M_2^{2n},\omega_2)$,

2. (**conformality**) $c(M,\lambda\omega) = |\lambda| c(M,\omega)$,

3. (**nontriviality**) $c(B^{2n}(1),\omega_0) > 0$ and $c(Z^{2n}(1),\omega_0) < \infty$.

Not that the first axiom implies naturality: if (M_1^{2n},ω_1) and (M_2^{2n},ω_2) are symplectomorphic then $c(M_1^{2n},\omega_1) = c(M_2^{2n},\omega_2)$. The deep link between the Gromov width and the idea of symplectic capacity lies in nontriviality axiom: it prevents the volume of M from being a capacity. In fact it means that capacities are 2-dimensional invariants as well as Gromov width is. The following theorem shows the relation we mentioned above.

Theorem 1.14. *The existence of a symplectic capacity c such that*

$$c(B^{2n}(1),\omega_0) = c(Z^{2n}(1),\omega_0) = \pi \tag{1.9}$$

is equivalent to Gromov's nonsqueezing theorem. Moreover the smallest of all these capacities is the Gromov width c_G.

Proof. Note that equation (1.9) is equivalent to the following

$$c(B^{2n}(r), \omega_0) = c(Z^{2n}(r), \omega_0) = \pi r^2$$

since for every subset $U \subset \mathbb{R}^{2n}$ there exist a symplectomorphism

$$\psi : (\lambda U, \omega_0) \to (U, \lambda^2 \omega_0)$$

In fact it is given by $x \mapsto \frac{1}{\lambda} x$ and the claim follows from the conformality axiom.
So now assuming Gromov's nonsqueezing theorem we prove that the Gromov width c_G is indeed a capacity which satisfies (1.9).
Monotonicity axiom holds because composition of symplectomorphisms is a symplectomorphism. In order to prove the second axiom we prove that to every embedding

$$\varphi : (B^{2n}(r), \omega_0) \to (M, \lambda\omega)$$

corresponds an embedding

$$\hat{\varphi} : (B^{2n}(\frac{r}{\sqrt{|\lambda|}}), \omega_0) \to (M, \omega)$$

and conversely so that by definition of c_G we get the assertion. If φ is given then we have

$$\varphi^*(\omega) = \frac{1}{\lambda}\omega_0$$

Now consider the symplectomorphism $\psi : (B^{2n}(\frac{r}{\sqrt{|\lambda|}}), \omega_0) \to (B^{2n}(r), \frac{1}{|\lambda|}\omega_0)$ we constructed at the beginning of the proof. Then if $\lambda > 0$ the embedding we are looking for is $\hat{\varphi} = \varphi \circ \psi$. If $\lambda < 0$ then the embedding is given by $\hat{\varphi} = \varphi \circ \psi \circ \psi_0$ where ψ_0 is

$$\psi_0 : \left(B^{2n}\left(\frac{r}{\sqrt{|\lambda|}}\right), \omega_0\right) \to \left(B^{2n}\left(\frac{r}{\sqrt{|\lambda|}}\right), -\omega_0\right), \quad (x, y) \mapsto (-x, y)$$

We prove now $c_G(B^{2n}(r), \omega_0) = c_G(Z^{2n}(r), \omega_0) = \pi r^2$. Note that, since a symplectomorphism is volume preserving, we must have $R \leq r$ for the existence of an embedding $\varphi : (B^{2n}(R), \omega_0) \to (B^{2n}(r), \omega_0)$. On the other hand the identity is a symplectomorphism and thus $c_G(B^{2n}(r), \omega_0) = \pi r^2$. The equality $c_G(Z^{2n}(r), \omega_0) = \pi r^2$ comes directly from Gromov's nonsqueezing theorem.

Conversely, let c be a capacity that satisfies (1.9), then the nonsqueezing theorem follows from monotonicity axiom.

For the last part of the theorem consider any capacity c satisfying (1.9) and an embedding

$$\varphi : (B^{2n}(r), \omega_0) \to (M, \omega)$$

Monotonicity axiom implies that $\pi r^2 = c(B^{2n}(r), \omega_0) \leq c(M, \omega)$ and taking the supremum we get the claim. □

The Gromov width is not the only interesting capacity, the most used is the Hofer-Zender capacity but we will not go further in this direction since we are interested only in Gromov width of manifolds. Anyway we now present some properties of Gromov width which hold true for all capacities. In particular we can try to extend the definition of capacities including subsets of $\mathbb{R}^{2n}$ in analogy with the fact that it can be easily done for the Gromov width. In order to do so we need the following definition: a *symplectic embedding* $\psi : A \to \mathbb{R}^{2n}$ of an *arbitrary subset* A of $\mathbb{R}^{2n}$ (with symplectic form inherited from $\mathbb{R}^{2n}$) is a map which extends to a symplectic embedding in a neighbourhood of A.

Definition 1.15. An **intrinsic symplectic capacity** c assigns to every subset $A \subset \mathbb{R}^{2n}$ a number $c(A) \in [0, \infty]$ such that the following hold

1. (**monotonicity**) If there exist a symplectic embedding $\psi : A \to \mathbb{R}^{2n}$ such that $\psi(A) \subset B$ then $c(A) \leq c(B)$,
2. (**conformality**) $c(\lambda A) = \lambda^2 c(A)$,
3. (**nontriviality**) $c(B^{2n}(1), \omega_0) > 0$ and $c(Z(1), \omega_0) < \infty$.

We already know that c_G satisfies these three axioms. We call these capacities intrinsic in order to underline the fact that $c(A)$ does not depend on how A is embedded in $\mathbb{R}^{2n}$ but only on the symplectic structure on A. We start with one of the most studied subsets of $\mathbb{R}^{2n}$, the ellipsoids, but we present only the basic results. An ellipsoid E is given by:

$$E = \left\{ x \in \mathbb{R}^{2n} | \sum_{i,j=1}^{2n} a_{ij} x_i x_j \leq 1 \right\}$$

but we know that, by a linear symplectomorphism, it can always be put in the form

$$E = \left\{ z \in \mathbb{C}^n | \sum_{j=1}^{n} \left| \frac{z_j}{r_j} \right|^2 \leq 1 \right\}$$

where $r = (r_1; \ldots; r_n)$, with $r_1 \leq \cdots \leq r_n$, is called the spectrum of E.
we can easily see that since

$$B^{2n}(r_1) \subset E \subset Z^{2n}(r_1)$$

we must have $c(E) = \pi r_1^2$ for every symplectic capacity which satisfies (1.9). This holds true in general: for every subset U of $\mathbb{R}^{2n}$ and every capacity c satisfying (1.9) if $B^{2n}(r) \subset E \subset Z^{2n}(r)$ then we have $c(U) = \pi r^2$.

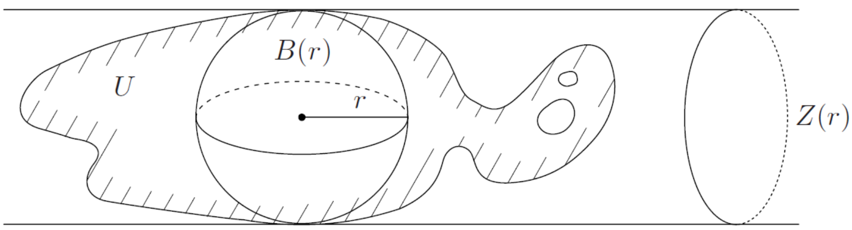


Figure 1.3: $B^{2n}(r)$ contained in U embedded in $Z^{2n}(r)$

Then next proposition follows directly from what we said:

Proposition 1.16. *Assume E and F are two ellipsoids and $\varphi : E \to F$ is a symplectic embedding, then*

$$r_1(E) \leq r_1(F)$$

Our last result on ellipsoids is the following

Proposition 1.17. *There exist a symplectomorphism*

$$\varphi : B^2(r_1) \times B^2(r_2) \to B^2(s_1) \times B^2(s_2)$$

if and only if $r_1 = s_1$ *and* $r_2 = s_2$.

Proof. Since $r_1 = r_2$ we can use the symplectomorphism φ to define a symplectic embedding:

$$B^4(r_1) \to B^2(r_1) \times B^2(r_2) \to B^2(s_1) \times B^2(s_2) \to Z(s_1)$$

By monotonicity of c we conclude $r_1 \leq s_1$. Applying the same argument to φ^{-1} we get $r_1 \geq s_1$. Now, since a symplectomorphism is volume preserving, the last equality follows from $r_1 r_2 = s_1 s_2$. $\square$

In order to conclude this section we introduce a result that explains in which sense the capacities are 2-dimensional invariants:

Proposition 1.18. *Assume* $\Omega \subset \mathbb{R}^{2n}$ *is an open bounded nonempty set and* $W \subset \mathbb{R}^{2n}$ *is a linear subspace with codim*$W = 2$ *and consider the cylinder* $\Omega + W$. *Then*

i) $c(\Omega + W) = \infty$ *if* $W^{\perp}$ *is isotropic*

ii) $0 < c(\Omega + W) < \infty$ *if* $W^{\perp}$ *is not isotropic*

1.3 Hermitian symmetric spaces

In this section we introduced the class of spaces we are going to deal with and state their main properties. Let us start directly with the definition:

Definition 1.19. Let M be a connected complex manifold with a Hermitian structure. M is said to be an Hermitian symmetric space if each point $p \in M$ is an isolated fixed point for an involutive holomorphic isometry s_p of M.

From now on we will write in short HSS meaning Hermitian symmetric space. The group $I(M)$, namely the group of isometries of M, has a structure of Lie group compatible with the open-compact topology and is a Lie

transformation group of M. Moreover, the group of holomorphic isometries of M, which we denote with $A(M)$, is a closed subgroup of $I(M)$ and thus a Lie transformation group of M itself. The group $A(M)$ and its identity component G act transitively on M.
Now choose a point $p \in M$ and let K be the subgroup of G leaving p fixed. It can be proved that M is diffeomorphic to G/K under the map $g(p) \mapsto gK$ where $g \in G$.
As a first consequence of this definition we get that any Hermitian symmetric space M is a Kähler manifold.
The following proposition gives a rough idea of the structure of a HSS.

Proposition 1.20. *Every HSS admits a Hermitian isometry with a space $M_0' \times M_1 \times \cdots \times M_k$ where M_0' is the quotient of a complex euclidean space by a discrete group of pure translations and the M_i are irreducible simply connected HSS.*

Proof. We consider the de Rham decomposition of the universal covering $\tilde{M}$ of a Hermitian symmetric space M:

$$\tilde{M} = M_0 \times M_1 \times \cdots \times M_k$$

where we know that $M_1 \ldots M_k$ are irreducible (not euclidean and not locally isomorphic to a product of lower dimensional manifolds) and M_0 is an Euclidean space. Moreover $\tilde{M}$ is symmetric (Kähler) if and only if each M_i is symmetric (Kähler). Thus we can conclude that $\tilde{M} \cong M_0 \times M_1 \times \cdots \times M_k$ is a Hermitian isometry. Now if $\pi : \tilde{M} \to M$ is the universal covering there is a unique complex structure and a unique Hermitian metric on $\tilde{M}$ such that π is locally a Hermitian isometry. That makes $\tilde{M}$ a HSS since the symmetries of M lift. Now $M = \tilde{M}/\Gamma$ where Γ is a discontinuous group of Hermitian isometries and then M_0 is complex and the M_i are irreducible simply connected HSS. Since Γ preserves each M_i ([24]) it acts as a group of pure translations on M_0 and acts trivially on each M_i. □

Referring to the previous de Rham decomposition we say that the space M is

1. Euclidean if $\tilde{M} = M_0$,

2. Irreducible if $\tilde{M} = M_1$,
3. Strictly non-Euclidean if $\tilde{M} = M_1 \times \cdots \times M_k$,
4. of compact type if $\tilde{M} = M_1 \times \cdots \times M_k$ and each M_i is compact,
5. of noncompact type if $\tilde{M} = M_1 \times \cdots \times M_k$ and each M_i is noncompact.

We will deal with HSS that fall into the third case. Note that if M is strictly non-Euclidean then it always holds true that $\pi_1(M) = 0$.
The last two cases are closely related to each other. In fact there is a duality between HSS of compact type (in short $HSSCT$) and HSS of noncompact type (in short $HSSNT$) that will play a key role hereafter. We explain now how this duality is expressed.
We start with a $HSSCT$ M^*. Recall that $M^* \cong G/K$ where G is the identity component of the group of Hermitian isometries $A(M^*)$ of M^* and K the isotropy group at $p \in M^*$. Denote as always the symmetry at p with s_p.
The Lie algebra $\mathfrak{g}$ of G can be decomposed with respect to the (± 1)-eigenspaces of the adjoint $ad(s_p)$ of s_p: $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$. That gives another algebra

$$\mathfrak{g}_0 = \mathfrak{k} + \mathfrak{m}_0 \qquad \mathfrak{m}_0 = i\mathfrak{m}$$

that has the same complexification as $\mathfrak{g}$. Passing to the group level we get the $HSSNT$ $M = G_0/K$. One can easily see that applying this construction to M the result is M^*. We will then say that:

$$M \text{ is the (noncompact) dual of } M^*$$
$$M^* \text{ is the (compact) dual of } M.$$

We will always denote a $HSSCT$ (resp. $HSSNT$) by M^* (resp. M). The duality yields some interesting relations between $HSSCT$ and their duals. In particular we can always holomorphically embed a $HSSNT$ in its compact dual via the Borel embedding. In order to present this link let us introduce some notation:

- $\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} + \mathfrak{m}_{\mathbb{C}}$ is the complexification of $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$
- $G_{\mathbb{C}}$ is the complex Lie group associated to $\mathfrak{g}_{\mathbb{C}}$

- $z \in \mathfrak{k}$ is a central element such that $J = ad(z)_{|\mathfrak{m}_{\mathbb{C}}}$ is the complex structure on M and M^*)
- $\mathfrak{m}_{\mathbb{C}}^{\pm}$ are the (± 1)-eigenspaces of J
- $\mathfrak{p} = \mathfrak{k}_{\mathbb{C}} + \mathfrak{m}_{\mathbb{C}}^{-}$ is a parabolic subalgebra of $\mathfrak{g}_{\mathbb{C}}$ that is the sum of the nonnegative eigenspaces of $ad(iz) : \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}$
- P is the Lie group associated to $\mathfrak{p}$

We can now prove the

Theorem 1.21 (Borel embedding)**.** *G is transitive on the complex coset space $G_{\mathbb{C}}/P$ with isotropy group $G \cap P = K$; thus*

$$M^* = G_{\mathbb{C}}/P$$

Moreover if $p = 1 \cdot P \in G_{\mathbb{C}}/P$ then $G_0 \cap P = K$ and thus $gP \mapsto g(p)$ embeds M holomorphically as an open G_0-orbit.

Proof. Since $\mathfrak{g} \cap \mathfrak{p} = \mathfrak{k}$ we have $\dim G(p) = \dim \mathfrak{g} - \dim \mathfrak{k} = \dim \mathfrak{m} = \dim \mathfrak{m}_{\mathbb{C}}^{+} = \dim G_{\mathbb{C}}/P$ where dim denotes the dimension over $\mathbb{R}$. Thus $G(p)$ is open in $G_{\mathbb{C}}/P$ and the same holds for $G_0(p)$. Now since G is compact also $G(p)$ is compact and then closed (as well as open) in the connected space $G_{\mathbb{C}}/P$. Hence we get $G_{\mathbb{C}}/P = G(p)$. Now $gK \mapsto g(p)$ gives a complex analytic covering space $M^* \to G_{\mathbb{C}}/P$. This endows $G_{\mathbb{C}}/P$ with the structure of $HSSCT$ implying it is simply connected. Thus we conclude $G_{\mathbb{C}}/P \cong M^*$. Similarly $M = G_0/K \to G_0(p)$ is a complex analytic diffeomorphism. □

This does not end the list of relations between duals. The next theorem shows that we can regard M as an open bounded (symmetric) domain in $\mathfrak{m}_{\mathbb{C}}^{+}$. This will be studied in next chapter in a more convenient context thus we do not linger on proving this theorem or discussing the consequences here.

Theorem 1.22 (Harish-Chandra Embedding)**.** *The map*

$$\xi : \mathfrak{m}_{\mathbb{C}}^{+} \to M^* = G_{\mathbb{C}}/P \quad \textit{given by} \quad \xi(p) = \exp(p)P$$

is a complex analytic diffeomorphism of $\mathfrak{m}^+_{\mathbb{C}}$ *onto an open dense subset of* M^* *that contains* M*. Furthermore* $\xi^{-1}(M) = \Omega$ *is an open bounded symmetric domain in* $\mathfrak{m}^+_{\mathbb{C}}$*.*

As a consequence the space $\mathfrak{m}^+_{\mathbb{C}}$ inherits the Kähler structure on M^* as we will see in next chapter. On the other hand, since $m^+_{\mathbb{C}}$ is a complex vector space, it is a Kähler manifolds with the flat form ω_0. In other words we are saying that $m^+_{\mathbb{C}}$ is a Kähler manifold with respect to both the forms ω_{FS} (coming from M^*) and ω_0 (coming from $\mathbb{C}^n = m^+_{\mathbb{C}}$). Even if we are going to study these spaces later we state here some of the structure on a bounded symmetric domain.
So let $\Omega \subset \mathbb{C}^n$ be a bounded symmetric domain and denote with $H^2(\Omega)$ the set of functions of $L^2(\Omega)$ which are holomorphic in Ω.
This is actually a complete Hilbert subspace of $H^2(\Omega)$ with the inner product

$$\big(f|g\big) = \int_\Omega f(z)\overline{g(z)}d\mu(z)$$

where $d\mu(z)$ is the Lebesgue measure on $\mathbb{R}^{2n}$. Now for every $w \in \Omega$, by Riesz representation theorem, there exists $K_w \in H^2(\Omega)$ such that

$$f(w) = \big(f|K_w\big) \qquad \forall f \in H^2(\Omega)$$

Definition 1.23. The Bergman kernel K_Ω (or shortly K) of Ω is the function $K : \Omega \times \Omega \to \mathbb{C}$ defined by

$$K(z, w) := K_w(z) = \big(K_w|K_z\big)$$

Furthermore for any complete orthonormal system $\{\varphi_j\}$, applying the evaluation at z of the Hilbert space we see that

$$K(z, w) = \sum_{j=0}^{\infty} \varphi_j(z)\overline{\varphi_j(w)}$$

Another property of the Bergman kernel is that it is invariant under isomorphism: if $F : \Omega \to \Omega'$ is holomorphic with holomorphic inverse then

$$K_{\Omega'}(F(z), F(w)) j_F(z) \overline{j_F(w)} = K_\Omega(z, w)$$

where j_F is the complex Jacobian of F. However the main result about the Bergman kernel is the following

Theorem 1.24. *Let Ω be a bounded symmetric domain and K its Bergman kernel. Then the matrix*

$$g_{ij}(z) = \frac{\partial^2}{\partial z_i \partial \overline{z_j}} \log K(z, z)$$

defines an invariant Kähler metric on Ω which is called Bergman metric.

Here, as above, invariant means that $|dF(X)| = |X|$ if $F : \Omega \to \Omega'$ is an isomorphism. Moreover next theorem shows that the correspondence between bounded symmetric domains and $HSSNT$ works in both ways

Theorem 1.25. *Each bounded symmetric domain Ω is, when equipped with the Bergman metric, a $HSSNT$. In particular Ω is simply connected.*

We can say something also in the case M^* is a $HSSCT$: it can be proved that every Hermitian symmetric space of compact type can be holomorphically embedded in $\mathbb{C}P^d$ for some d which makes M^* a complex projective variety. Thus the Kähler form on M^* is induced by the Fubini-Study form of $\mathbb{C}P^d$. With an abuse of notation we will indicate with ω_{FS} this form on M^*. From now on, when we say that M^* is a $HSSCT$ we mean it equipped with the form ω_{FS} normalized so that $\omega_{FS}(A) = \pi$ where $[A]$ is the generator of $H_2(M^*, \mathbb{Z})$. We will see a realization of M^* as projective variety in next chapter in the language of Jordan triple systems.
Now let us come back to the distinction between $HSSCT$ and their non-compact duals. Recall that every strictly non-Euclidean HSS is a product of irreducible $HSSCT$ and $HSSNT$. These irreducible spaces have very nice characterization:

Theorem 1.26.

- *The irreducible $HSSNT$ are exactly the manifolds G/K where G is a connected noncompact simple Lie group with center $\{e\}$ and K has nondiscrete center and is a maximal compact subgroup of G.*

- *The irreducible $HSSCT$ are exactly the manifolds G/K where G is a connected compact simple Lie group with center $\{e\}$ and K has nondiscrete center and is a maximal connected proper subgroup of G.*

Furthermore in 1935 H. Cartan classified the irreducible bounded symmetric domains (thus irreducible $HSSNT$ and their compact duals) into four classical families and two exceptional cases. This is Cartan classification:

- Type $I_{k,n}$ $(k \leq n)$: the domain of $k \times n$ matrices $Z \in \mathbb{C}^{kn}$ satisfying the condition $Id - Z\overline{Z}' > 0$. It has real dimension $2kn$.

- Type II_n $(n \geq 1)$: the domain of $n \times n$ symmetric matrices satisfying the condition $Id - Z\overline{Z} > 0$. It has real dimension $n(n+1)$.

- Type III_n $(n \geq 2)$: the domain of $n \times n$ skew-symmetric matrices satisfying the condition $Id + Z\overline{Z} > 0$. It has real dimension $n(n-1)$.

- Type $IV_n (n > 2)$: the so-called Lie ball of $\mathbb{C}^n$ that is the domain of $z \in \mathbb{C}^n$ such that

  $$\overline{z}'z < 1, \qquad 1 + |z'z| - 2\overline{z}'z > 0$$

  It has real dimension $2n$.

- Type V: exceptional domain of dimension 16

- Type VI: exceptional domain of dimension 27

Here and throughout the thesis $\overline{Z}'$ will denote the conjugate transpose of Z. Once we know this classification we can construct all $HSSNT$ and thus, knowing the dual classification, all $HSSCT$. For example the compact dual of the first domain is the Grassmannian of k-planes in $\mathbb{C}^{n+k}$ and the compact dual of the domain of type IV is the complex projective quadric Q_n in $\mathbb{C}P^{n+1}$ i.e.

$$Q_n = \left\{ [z_0 : \ldots : z_{n+1}] \in \mathbb{C}P^{n+1} \middle| \sum_{i=0}^{n+1} z_i^2 = 0 \right\}$$

In order to give a deeper understanding of what we presented in this section we investigate the simplest possible nontrivial case: the domain $\Omega = I_{1,2}$ and its compact dual M^* which is nothing but $\mathbb{C}P^1$.

We consider the Riemann sphere $S^2(1) = \mathbb{C}P^1$. In this case the groups are:

$$G_{\mathbb{C}} = \left\{ \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| ad - cb = 1 \right\}$$

$$P = \left\{ \pm \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \middle| ad = 1 \right\}$$

We have $M^* = G_{\mathbb{C}}/P$ where $G_{\mathbb{C}}$ acts by

$$\pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \mapsto (az + b)/(cz + d)$$

Now passing to the noncompact dual the groups are

$$G_0 = \left\{ \pm \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \middle| |a|^2 - |b|^2 = 1 \right\}$$

$$K = \left\{ \pm \begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix} \middle| |a|^2 = 1 \right\}$$

Under the action of G_0, M^* decomposes into two open orbits: the upper hemisphere ($G_0(0) = M$) and the lower hemisphere ($G_0(\infty)$) and a closed one that is the equator ($G_0(i)$).

Now $\mathfrak{m}_{\mathbb{C}}^+ = \mathbb{C} = M^* \setminus \{\infty\}$ is embedded in M^* via the stereographic projection. Hence the realization of M as a bounded symmetric domain is the open disc in the complex line $\mathbb{C}$: $M = \{z \in \mathbb{C} : \ |z|^2 < 1\}$.

Further material on HSS from the Lie theoretic point of view can be found for example in [5, 25, 26].

Chapter 2

Geometry of Hermitian symmetric spaces

This chapter is dedicated to the study of the geometry of Hermitian symmetric spaces of compact and noncompact type. The theory introduced here will be of great importance in the last chapter of the thesis. In the first section we establish a correspondence between HSS and Jordan triple systems which give us a useful language we will use in the remainder of the thesis. In section 2 we present a recent result which gives a deep relation between Jordan triple systems and $HSSNT$. In the last part of the chapter we explain the work [12] of Loi-Mossa-Zuddas where they compute the Gromov width of Hermitian symmetric spaces of compact and noncompact type. This last part is the key ingredient of our work.

2.1 Jordan triple systems

An alternative approach to the Lie theoretic methods for the study of Hermitian symmetric spaces is provided by Jordan triple systems (JTS). In particular there is a one-to-one correspondence between bounded symmetric domains and Hermitian positive Jordan triple systems ($HPJTS$). At the end of the section we will also present a realization of M^* as a complex projective variety.
We present here some basic facts about JTS that can be found in [20, 3].

Definition 2.1. A Hermitian Jordan triple system is a (finite dimensional) complex vector space $\mathcal{V}$ equipped with an antilinear involution $z \mapsto \bar{z} : \mathcal{V} \to \mathcal{V}$ (called conjugation) and a trilinear map $\{,,\} : \mathcal{V} \times \mathcal{V} \times \mathcal{V} :\to \mathcal{V}$ called triple product such that

$$\{u, v, w\} = \{w, v, u\}$$

$$\{x, y, \{u, v, w\}\} - \{u, v, \{x, y, w\}\} = \{\{x, y, u\}, v, w\} - \{u, \{v, x, y\}, w\}$$

We will consider only the case of simple JTS that means $\mathcal{V}$ is not the sum of two nontrivial subsystem with component-wise triple product. On a JTS we can define the operators

$$D(u, v)w = \{u, v, w\}$$
$$Q(u, w)v = \{u, v, w\}$$
$$Q(u) = \frac{1}{2}Q(u, u)$$

$$B(u, v) = Id - D(u, v) + Q(u)Q(v) \tag{2.1}$$

Note that they depend only on the triple product on $\mathcal{V}$. We will then say that $\mathcal{V}$ is a Hermitian positive Jordan triple system if the product

$$(u|v) = \frac{1}{g}\mathrm{tr}D(u, \overline{v}) \tag{2.2}$$

(where g is defined by (2.4) below) is positive definite on $\mathcal{V}$.
We can then state the correspondence between $HPJTS$ and $HSSNT$:

Theorem 2.2. *To every $HPJTS$ $\mathcal{V}$ is associated a $HSSNT$ realized as circled bounded domain $\Omega_{\mathcal{V}}$ centered in $0 \in \mathcal{V}$. It is the connected component containing the origin of the set of all $u \in \mathcal{V}$ such that $B(u, u)$ is positive defined with respect to $(\cdot|\cdot)$.*
Conversely the $HPJTS$ can be recovered from $\Omega_{\mathcal{V}}$ as the tangent space at the origin $\mathcal{V} = T_0\Omega_{\mathcal{V}}$ with the triple product given by

$$\{u, v, w\} = -\frac{1}{2}(R_0(u, v)w + J_0R_0(u, J_0v)w) \tag{2.3}$$

where R_0 (resp. J_0) is the curvature tensor of the Bergman metric (resp.

the complex structure) of $\Omega_{\mathcal{V}}$ *evaluated in the origin.*

We can give another and more useful way of describing $\Omega_{\mathcal{V}}$ in $\mathcal{V}$. In order to do this we need to focus on $HPJTS$. We start giving a particular decomposition of Hermitian JTS. An element $c \in \mathcal{V}$ is called tripotent if $\{c, \overline{c}, c\} = 2c$. If $c \in \mathcal{V}$ is a tripotent then the operator $D(c, \overline{c})$ (which is self-adjoint with respect to $(\cdot|\cdot)$) has its eigenvalues in $\{0, 1, 2\}$ and we have

$$\mathcal{V} = V_0(c) \oplus V_1(c) \oplus V_2(c)$$

This is called the Pierce decomposition of $\mathcal{V}$ with respect to c. We will call two tripotents c_1, c_2 orthogonal if $D(c_1, \overline{c_2}) = 0$.
Consider now a family of mutually orthogonal tripotents $(c_1, \ldots, c_p)$. Such a family is called a frame if it is maximal. It turns out that all frames have the same number r of elements and this number is called the rank of V.
Let $(c_1, \ldots, c_p)$ be any family of tripotents, then $\big(D(c_j, \overline{c_j})\big)$ is a family of commutative self-adjoint operators, giving rise to the simultaneous Pierce decomposition

$$\mathcal{V} = \sum_{j=0}^{p} \sum_{i=0}^{j} V_{ij}$$

where

$$V_{ij} = \{u \in V | \;\; D(c_k, \overline{c_k})u = (\delta_i^k + \delta_j^k)u; \;\; k = 1, \ldots, p\}$$

When $(c_1, \ldots, c_r)$ is a frame the simultaneous Pierce decomposition has nice properties:

1. $V_{00} = 0$
2. $V_{ii} = \mathbb{C}c_i$ for $i = 1, \ldots, r$
3. All V_{ij} have the same dimension $a = \dim V_{ij}$ for $1 \leq i < j \leq r$
4. All V_{0j} have the same dimension $b = \dim V_{0j}$ for $1 \leq j \leq r$

This leads us to the definition of another invariant of V: the genus g, given by

$$g = 2 + a(r - 1) + b \tag{2.4}$$

We want now to explain how one can regard a bounded symmetric domain in $\mathcal{V}$. Consider an element $u \in \mathcal{V}$. It has a unique spectral decomposition

$$u = \lambda_1 c_1 + \cdots + \lambda_p c_p$$

where $(c_1, \ldots, c_p)$ is a family of mutually orthogonal tripotents and $\lambda_1 > \lambda_2 > \cdots > \lambda_p > 0$. The map $u \mapsto \lambda_1$ is a norm $\| \cdot \|_{max}$ on V called the spectral norm.

Theorem 2.3. *Let $\Omega_{\mathcal{V}}$ be the HSSNT associated to V. Then*

$$\Omega_{\mathcal{V}} = \{u \in V | \|u\|_{max} < 1\}$$

An element u is called regular if $p = r$ in its spectral decomposition. The set of regular points is dense in $\mathcal{V}$. Thus let again $u = \lambda_1 c_1 + \cdots + \lambda_r c_r$ be the spectral decomposition of a regular element $u \in \mathcal{V}$. It is not hard to prove that

$$B(u, \overline{u})v = (1 - \lambda_i^2)(1 - \lambda_j^2)v \tag{2.5}$$

if $v \in V_{ij}$. From this formula we easily get the following

$$\det B(u, \overline{u}) = \left(\prod_{i=1}^{r} (1 - \lambda_i^2) \right)^g$$

In particular one can recover the Bergman form on $\Omega_{\mathcal{V}}$ as

$$\omega_{Berg} = -\frac{i}{2\pi} \partial\overline{\partial} \log \det B$$

The real polynomial $\prod_{i=1}^{r} (1 - \lambda_i^2)$ is called the generic norm and indicated with $m(u, \overline{u})$. It can be extended to a complex polynomial $m(u, v)$ on $\mathcal{V} \times \mathcal{V}$ which is also called the generic norm and has the expansion

$$m(u, v) = 1 - m_1(u, v) + \cdots + (-1)^j m_j(u, v) + \cdots + (-1)^r m_r(u, v)$$

where the m_j's are nondegenerate polynomials, homogeneous of bidegree (j, j). We are now able to define the hyperbolic form we introduced in the previous chapter.

$$\omega_{hyp} := -\frac{i}{2\pi}\partial\overline{\partial}\log m = \frac{\omega_{Berg}}{g} \tag{2.6}$$

Moreover by means of the generic norm we can define the flat Kähler form on $\mathcal{V}$

$$\omega_0 := \frac{i}{2\pi}\partial\overline{\partial}m_1 \tag{2.7}$$

Since we are considering only the simple JTS we can rewrite $\omega_0 = \frac{i}{2\pi}\partial\overline{\partial}\mathrm{tr}D$. Note that in the rank 1 case it is the standard Euclidean form on $\mathcal{V} = \mathbb{C}^n$. The next proposition explains the structure of the generic norm:

Proposition 2.4. *Let $\mathcal{V}$ be a simple Hermitian JTS and $m(u, v)$ its generic norm. There exist maps $\sigma_j : \mathcal{V} \to V^{(j)}$ for $J = 0, \ldots, r$ with the following properties:*

1. *$V^{(0)} = \mathbb{C}$, $V^{(1)} = \mathcal{V}$, $V^{(2)}, \ldots, V^{(r)}$ are finite dimensional complex vector spaces with conjugation $z \mapsto \overline{z}$ and inner product $(:)$.*

2. *$\sigma_0 = 1$, $\sigma_1 = Id$, $\sigma_2, \ldots, \sigma_r$ are homogeneous polynomial maps of bidegree j such that $\sigma_j(\overline{u}) = \overline{\sigma_j(u)}$ and $\sigma_j(V)$ spans $V^{(j)}$.*

3. *the identity*

$$m(u, v) = 1 - (\sigma_1 u, \sigma_1 v) + \cdots + (-1)^j(\sigma_j u, \sigma_j v) + \cdots + (-1)^r(\sigma_r u, \sigma_r v)$$

holds in $\mathcal{V} \times \mathcal{V}$.

The only prove of proposition 2.4 known to the author is a case by case verification which we do not report here. We want now to introduce the compactification of $\mathcal{V}$. The construction of the canonical compactification was presented by Loos ([13]) but we would like to present it in an equivalent form which is based on the theory exhibited above.
Then let us state the theorem that constructs the compactification of $\mathcal{V}$ as a complex projective variety which is isomorphic to the compact dual of the $HSSNT$ associated to $\mathcal{V}$.

Theorem 2.5. *Let $\mathcal{V}$ be a simple Hermitian JTS of rank r and $\sigma_1, \ldots, \sigma_r$*

as in proposition 2.4. Let

$$W = \mathbb{C} \oplus V^{(1)} \oplus \cdots \oplus V^{(r)}$$

and $\sigma : \mathcal{V} \to \mathbb{P}(W)$ *be defined by*

$$\sigma(v) = [1 : \sigma_1 v : \ldots : \sigma_r v]$$

Then the closure of $\sigma(\mathcal{V})$ *in* $\mathbb{P}(W)$ *is an algebraic submanifold* X *which is isomorphic to the compact dual of* $\Omega_{\mathcal{V}}$ *and* $\sigma : \mathcal{V} \to \mathbb{P}(W)$ *is isomorphic to the canonical compactification.*

The previous theorem implies that, given any $HSSNT$, its compact dual M^* can be embedded in $\mathbb{C}P^d$ for some $d > 0$. It means that M^* is an Hermitian Symmetric space of compact type with the Kähler form induced by the Fubini-Study form on $\mathbb{C}P^d$. That is what we run over in the previous chapter.

2.2 The symplectic duality

In this section we present the work of Di Scala and Loi ([1]). In particular we are going to show that there is a symplectic duality Ψ between $\mathcal{V}$ and $\Omega_{\mathcal{V}}$ and that by the theory of HSS it extends to a symplectic embedding from $\Omega_{\mathcal{V}}$ to $M^* \setminus Y_p$ where Y_p is the cut locus of $p \in M^*$.
We will prove this theorem in the case of $HSSNT$ of classical type even if in ([1]) is also provided a proof which holds for all $HSSNT$.
Let us start with some result which will be needed in the proof of the theorem. Note first that under the identification $m^+_{\mathbb{C}} \cong \mathcal{V}$ one can endow $\mathcal{V}$ with the restriction of the Fubini-Study form on M^*. It can be proved that this restriction is written as

$$\omega_{FS} = \frac{i}{2\pi} \partial\bar{\partial} \log \det m^* \tag{2.8}$$

where $m^*(u, u) = m(u, -u) = 1 + \sum_j m_j(x, x)$.
The next result is of great importance for the proof.

Proposition 2.6. *Let $(M, 0)$ be a HSSNT with center 0 and let $\mathcal{V}$ be the associated HPJTS. Then there exist a one-to-one correspondence between (complete) complex totally geodetic submanifolds through the origin $(T, 0)$ and sub-HPJTS $\mathcal{T} \subset \mathcal{V}$ where $\mathcal{T}$ is the HPJTS associated to $(T, 0)$.*

Proof. It is known that there is a one-to-one correspondence between complex totally geodesic submanifolds of HSS through the origin and complex Lie triple system. This can be found for example in ([8]). Now from formula (2.3) and using the fact that

$$R_0(u, v)w = -\{u, \overline{v}, w\} + \{v, \overline{u}, w\}$$

it arises the one-to-one correspondence between complex Lie triple systems and sub-$HPJTS$ of $\mathcal{V}$. $\square$

We are now ready to state the main theorem in ([1]) in the form we are going to prove it. In the following theorem we will identify the $HSSNT$ M with its realization $\Omega_{\mathcal{V}}$ as circled bounded symmetric domain centered in the origin of $\mathcal{V}$.

Theorem 2.7 (Di Scala, Loi)**.** *Let M be a HSSNT with no exceptional factor in its de Rham decomposition. Let B be the Bergman operator on the associated HPJTS $\mathcal{V}$ defined in (2.1).*
Then the map

$$\Psi_M : M \to \mathcal{V}, \qquad z \mapsto B(z, z)^{-\frac{1}{4}} z,$$

has the following properties:

(D) Ψ_M *is a real analytic diffeomorphism and its inverse is given by:*

$$\Psi_M^{-1} : \mathcal{V} \to M, \qquad z \mapsto B(z, -z)^{-\frac{1}{4}} z;$$

(H) For any $(T, 0) \overset{i}{\hookrightarrow} (M, 0)$ complex and totally geodesic embedded submanifold $(T, 0)$ with $i(0) = 0$ one has

$$\Psi_{M|_T} = \Psi_T, \qquad \Psi_M(T) = \mathcal{T}$$

where $\mathcal{T}$ is the HPJT associated to T;

(I) *For any* $\tau \in K \subset I(M)$, *where* K *is the isotropy group at the origin, the following equality holds:*

$$\Psi_M \circ \tau = \tau \circ \Psi_T;$$

(S) Ψ_M *is a symplectic duality, i.e. the following hold*

$$\Psi_M^* \omega_0 = \omega_{hyp} \tag{2.9}$$

$$\Psi_M^* \omega_{FS} = \omega_0 \tag{2.10}$$

where ω_0 *on* M *is considered as the restriction of (2.7).*

Proof. The proof is divided in three parts. The first step consists in proving properties (D) and (S) in the special case of I_n which is the bounded symmetric domain of first type $I_{n,n}$. Then we will prove properties (H) and (I) for the four classical domains. Finally using proposition (2.6) and the second part we prove properties (D) and (S) for all classical $HSSNT$.
Step 1.(Proof of (D) and (S) for I_n)
We have already seen that

$$I_n = \{Z \in M_n(\mathbb{C}) | Id - Z\overline{Z}' > 0\}$$

The triple product making $\mathbb{C}^{n^2}$ a $HPJTS$ is given by:

$$\{U, V, W\} = U\overline{V}'W + W\overline{V}'U \quad U, V, W \in M_n(\mathbb{C}).$$

From this it is easy to calculate the Bergman operator:

$$B(U, V)W = (Id - U\overline{V}')W(Id - \overline{V}'U),$$

the hyperbolic form:

$$\omega_{hyp} = -\frac{i}{2\pi}\partial\overline{\partial} \log \det(Id - Z\overline{Z}')$$

and the map $\Psi_M : I_n \to M_n(\mathbb{C}) \cong \mathbb{C}^{n^2}$ is:

$$\Psi_M(Z) = (Id - Z\overline{Z}')^{-\frac{1}{2}} Z.$$

Thus we can calculate the explicit expression of the Fubini-Study (2.8) and the flat Kähler form (2.7) on $\mathbb{C}^{n^2}$:

$$\begin{aligned} \omega_{FS} &= \frac{i}{2\pi}\partial\overline{\partial}\log\det(Id + X\overline{X}'), \\ \omega_0 &= \frac{i}{2\pi}\partial\overline{\partial}\mathrm{tr}(X\overline{X}'). \end{aligned}$$

Now part (D) can be proved by verifying that the map:

$$\Phi_M : \mathbb{C}^{n^2} \to I_n, \quad X \mapsto (Id + X\overline{X}')^{-\frac{1}{2}} X.$$

is the inverse of Ψ_M and keeping in mind the equality:

$$X\overline{X}'(Id + X\overline{X}')^{\frac{1}{2}} = (Id + X\overline{X}')^{\frac{1}{2}} X\overline{X}'.$$

We are now ready to prove the property (S). Consider first the equality (2.9) and observe that we can write

$$\begin{aligned} \omega_{hyp} = -\frac{i}{2\pi}\partial\overline{\partial}\log\det(Id - Z\overline{Z}') &= \frac{i}{2\pi}\mathrm{d}\partial\log\det(Id - Z\overline{Z}') \\ = \frac{i}{2\pi}\mathrm{d}\partial\mathrm{tr}\log(Id - Z\overline{Z}') &= \frac{i}{2\pi}\mathrm{dtr}\partial\log(Id - Z\overline{Z}') \\ = -\frac{i}{2\pi}\mathrm{dtr}[\overline{Z}'(Id - Z\overline{Z}')^{-1}\mathrm{d}Z], \end{aligned}$$

where we used the decomposition $\mathrm{d} = \partial + \overline{\partial}$ and the identity $\log\det A = \mathrm{tr}\log A$. By substituting $X = (Id - Z\overline{Z}')^{-\frac{1}{2}}$ in the last equation we get

$$-\frac{i}{2\pi}\mathrm{dtr}\big[\overline{Z}'(Id - Z\overline{Z}')^{-1}dZ\big] = -\frac{i}{2\pi}\mathrm{dtr}\big[\overline{X}'\mathrm{d}X\big] + \frac{i}{2\pi}\mathrm{dtr}\big\{\overline{X}'\mathrm{d}\big[(Id - Z\overline{Z}')^{-\frac{1}{2}}\big]Z\big\}$$

Note that $-\frac{i}{2\pi}\mathrm{dtr}\big[\overline{X}'\mathrm{d}X\big] = \omega_0$ and the 1-form $\mathrm{tr}\Big\{\overline{X}'\mathrm{d}\big[(Id - Z\overline{Z}')^{-\frac{1}{2}}\big]Z\Big\}$ is exact being equal to $\mathrm{dtr}(\frac{C^2}{2} - \log C)$, where $C = (Id - Z\overline{Z}')^{-\frac{1}{2}}$.
Then equality (2.9) follows since ω_{hyp} equals ω_0 in the X-coordinates.

With the same arguments we can prove (2.10). Consider indeed

$$\begin{aligned}\omega_{FS} &= \frac{i}{2\pi}\partial\overline{\partial}\log\det(Id + X\overline{X}') = -\frac{i}{2\pi}\mathrm{d}\mathrm{tr}\partial\log(Id + X\overline{X}')\\ &= -\frac{i}{2\pi}\mathrm{d}\mathrm{tr}\big[\overline{X}'(Id + X\overline{X}')^{-1}\mathrm{d}X\big].\end{aligned}$$

Now substituting $Z = (Id + X\overline{X}')^{-\frac{1}{2}}$ we get

$$\begin{aligned}&-\frac{i}{2\pi}\mathrm{d}\mathrm{tr}\big[\overline{X}'(Id + X\overline{X}')^{-1}\mathrm{d}X\big]\\ &= -\frac{i}{2\pi}\mathrm{d}\mathrm{tr}\big(\overline{Z}'\mathrm{d}Z\big) + \frac{i}{2\pi}\mathrm{d}\mathrm{tr}\big\{\overline{Z}'\mathrm{d}\big[(Id + X\overline{X}')^{-\frac{1}{2}}\big]X\big\}\\ &= \omega_0 + \frac{i}{2\pi}\mathrm{d}^2\mathrm{tr}\Big(\log D - \mathrm{tr}\frac{D^2}{2}\Big) = \omega_0\end{aligned}$$

where $D = (Id + X\overline{X}')^{-\frac{1}{2}}$. This ends the first step of the proof.

Step 2.(Proof of (H) and (I) for classical domains)
From now on M will be a $HSSNT$ and $\mathcal{V}$ its associated $HPJTS$. Considering that the map Ψ_M depends only on the properties of the triple product $\{,,\}$, part (H) follows from proposition 2.6.
Hence we only need to prove (I). As we have seen in the first chapter the M can be regarded as the coset space $M = G/K$ where G is the connected component of $I(M)$ containing the origin and K is the isotropy group at the origin $0 \in M$. Cartan has proven ([19, p. 63]) that the group K consist entirely of linear transformations. In particular the Bergman operator of $\mathcal{V}$ is invariant under the action of K:

$$B(\tau u, \tau v)(\tau w) = \tau\big(B(u,v)(w)\big) \quad \forall \tau \in K,$$

which implies

$$B(\tau z, \tau z)^{-\frac{1}{4}}(\cdot) = \tau\big(B(z,z)^{-\frac{1}{4}}(\tau^{-1}(\cdot))\big) \quad \forall z \in M.$$

Thus (I) follows.
Step 3.(Proof of (D) and (S) for all classical domains)
It is known that every bounded symmetric domain can be embedded in I_N

for some N sufficiently large. This follows directly from the definition for $I_{k,n}$, II_n and III_n while the explicit embedding for IV_n can be found in ([16, p. 42]). Moreover we can always assume that the embedding brings the origin $0 \in M$ to the origin $0 \in I_n$.
Thus, by proposition 2.6, $\mathcal{V}$ is a sub-$HPJTS$ of $(\mathbb{C}^{n^2}, \{,,\})$. Hence the claim follows from property (H) and the fact that (D) and (S) hold true for I_n. □

2.3 Gromov width of Hermitian symmetric spaces

In this section we want to present the work of Loi, Mossa and Zuddas ([12]) where they computed the Gromov width and others symplectic capacities of Hermitian symmetric spaces of compact and noncompact type.
We will focus on the calculation of the Gromov width of $HSSCT$. As usually happens one computes the Gromov width of a manifold by giving upper and lower bounds. The lower bound is obtained by proving that the ball $(B^{2n}(1), \omega_0)$ can be embedded in any $HSSCT$. We will show how this embedding is constructed while we will only give the idea of how the upper bound is achieved.
Let (M^*, ω_{FS}) be a $2n$-dimensional $HSSCT$ and (M, ω_0) its noncompact dual regarded as a bounded symmetric domain in $(\mathcal{V}, (\cdot|\cdot)) \cong (\mathbb{C}^n, h_0)$ where $(\cdot|\cdot)$ was defined in (2.2) and h_0 is the canonical Hermitian product. Recall that throughout this thesis the canonical Kähler form ω_{FS} is normalized so that $\omega_{FS}(A) = \pi$ where $[A]$ is the generator of $H_2(M^*, \mathbb{Z})$.
Let us start showing that the ball $(B^{2n}(1), \omega_0)$ can be embedded in a bounded symmetric domain (M, ω_0). In order to do this consider now the spectral decomposition $v = \lambda_1 c_1 + \cdots + \lambda_r c_r$ of a regular point in $M \subset \mathcal{V}$.
The distance $\mathrm{d}(0, v) = (v|v)^{\frac{1}{2}}$ can be expressed in terms of the spectral decomposition of v. In fact the spaces V_{ij} in the Pierce decomposition of $\mathcal{V}$ with respect to the frame $(c_1, \ldots, c_r)$ are the eigenspaces for $D(v, v)$ with eigenvalues $(\lambda_i^2 + \lambda_j^2)$. As the subspaces V_{ii}, V_{ij} and V_{0j} for $0 < i < j$ have dimensions 1, a and b we obtain

$$\mathrm{tr} D(v, v) = g \sum_{i=1}^{r} \lambda_i^2$$

From the definition of the product $(\cdot|\cdot)$ easily follows that

$$\mathrm{d}(0,v) = \sqrt{\sum_{i=1}^{r} \lambda_i^2}$$

Recall that $M \subset \mathcal{V}$ is the set of point whose spectral norm is less than 1. Now from the identification $(\mathcal{V}, (\cdot|\cdot)) \cong (\mathbb{C}^n, h_0)$ and the fact that the set of regular points is dense we conclude that

$$(B^{2n}(1), \omega_0) \subset (M, \omega_0).$$

Recall that in the previous section we constructed a symplectomorphism $\Psi_M : M \to \mathcal{V}$ which, amongst other properties, satisfies $\Psi_M^* \omega_{FS} = \omega_0$. The form ω_{FS} appearing in the last equality is induced on $\mathcal{V}$ by its Harish-Chandra embedding $\xi : \mathcal{V} \to M^*$. Hence we actually get a symplectic embedding Φ_M of (M, ω_0) in (M^*, ω_{FS}). Thus we have proved that $c_G(M^*) \geq \pi$.
The upper bound $c_G(M^*) \leq \pi$ is obtained by estimating some (pseudo) symplectic capacities. We present here a rough idea of how it is done.
Loi, Mossa and Zuddas used the concept of k-pseudo symplectic capacity (due to Lu [14]) which is weaker than that of symplectic capacity and depends on the homology classes of the symplectic manifold. Formally if we denote with $\mathcal{C}(2n,k)$ the set of all tuples $(M, \omega; \alpha_1, \ldots, \alpha_k)$ consisting of a symplectic manifold (M, ω) and k nonzero homology classes $\alpha_i \in H_*(M; \mathbb{Q})$ then a k-pseudo symplectic capacity is a map $c^{(k)}$ from $\mathcal{C}(2n,k)$ to $[0, \infty]$ satisfying the following

1. (**pseudo monotonicity**) If there exist a symplectic embedding $\varphi : (M_1, \omega_1) \to (M_2, \omega_2)$ then, for any $\alpha_i \in H_*(M_1; \mathbb{Q})$, $i = 0, \ldots, k$,

$$c^{(k)}(M_1, \omega_1; \alpha_1, \ldots, \alpha_k) \leq c^{(k)}(M_2, \omega_2; \varphi_*(\alpha_1), \ldots, \varphi_*(\alpha_k))$$

2. (**conformality**) $c^{(k)}(M, \lambda\omega; \alpha_1, \ldots, \alpha_k) = |\lambda| c^{(k)}(M, \omega; \alpha_1, \ldots, \alpha_k)$,

3. (**nontriviality**) $c^{(k)}(B^{2n}(1), \omega_0; pt, \ldots, pt) = \pi = c^{(k)}(Z^{2n}(1), \omega_0; pt, \ldots, pt)$ where pt denotes the homology class of a point.

Lu defined two 2-pseudo symplectic capacities which he called of Hofer-Zehnder type and he proved that their values are always greater than or equal to that of the Gromov width.
These pseudo symplectic capacities of Hofer-Zehnder type are estimated by using other two pseudo symplectic capacities $GW(M, \omega; \alpha_1, \alpha_2)$ and $GW_0(M, \omega; \alpha_1, \alpha_2)$ which are defined in terms of Gromov-Witten invariants. These last invariants can be seen, roughly speaking, as functions counting, for an ω-tame almost complex structure J, the number of J-holomorphic curves of given genus representing an homology class $A \in H_2(M, \mathbb{Z})$ with k marked points p_i passing through cycles X_i representing k homology classes $\alpha_i \in H_*(M; \mathbb{Q})$.
Then the value of the pseudo symplectic capacities $GW(M, \omega; \alpha_1, \alpha_2)$ and $GW_0(M, \omega; \alpha_1, \alpha_2)$ is the infimum of the areas $\omega(A)$ of homology classes for which an associated Gromov-Witten invariant is nonzero.
Lu in [14] proved that the values of $GW(M, \omega; \alpha_1, \alpha_2)$ and $GW_0(M, \omega; \alpha_1, \alpha_2)$ are always greater than or equal to those of the 2-pseudo symplectic capacities of Hofer-Zehnder type.
Loi, Mossa and Zuddas proved the existence for any $HSSCT$ M^* with $\omega_{FS}(\mathbb{C}P^1) = \pi$ of two homology classes α and β such that the associated Gromov-Witten invariant is nonzero and deduced that $GW(M^*, \omega_{FS}; pt, \gamma) = GW_0(M^*, \omega_{FS}; pt, \gamma) = \pi$ where γ is either α or β.
Then, from what we said on these pseudo capacities, it follows that

$$c_G(M^*, \omega_{FS}) \leq \pi$$

One can prove that this result can be extended from irreducible to arbitrary $HSSCT$. In fact in the same article the authors proved the following

Theorem 2.8. *Let (M_i^*, ω_{FS}^i), $i = 1, \dots, r$, be irreducible $HSSCT$ of dimension $2n_i$ endowed with the canonical Kähler form ω_{FS}^i normalized so that $\omega_{FS}^i(A_i) = \pi$ where $[A_i]$ is the generator of $H_2(M_i^*, \mathbb{Z})$. Then*

$$c_G(M_1^* \times \cdots \times M_r^*, \omega_{FS}^1 \oplus \cdots \oplus \omega_{FS}^r) = \pi$$

Proof. The bound $c_G(M_1^* \times \cdots \times M_r^*, \omega_{FS}^1 \oplus \cdots \oplus \omega_{FS}^r) \leq \pi$ is obtained with the same estimates of pseudo symplectic capacities we mentioned above. The lower bound, instead derives from the product of the symplectic embedding we constructed above:

$$\times_{i=1}^{r} B^{2n_i}(1) \subset \times_{i=1}^{r} M_i \xrightarrow{\Phi_{M_1}\times\cdots\times\Phi_{M_r}} \times_{i=1}^{r} M_i^*$$

and from the natural inclusion

$$B^{2n_1+\cdots+2n_r}(1) \subset \times_{i=1}^{r} B^{2n_i}(1)$$

$\square$

Chapter 3

Minimal atlases for closed symplectic manifolds

In this chapter we focus on the work of Rudyak and Schlenk [21] on minimal atlases for closed symplectic manifolds. In the first section we introduce the setting and explain the results. We state the main theorem and give the sketch of the proof which is simple and elegant in the idea while we avoid to examine the realization of the proof because it is technical and does not give further information. In order to get a deeper understanding of the topic we discuss, in the second section, some examples that can be found in the paper. In the last section we study the case of the complex Grassmannian. In particular we show that the embeddings produced by Rudyak and Schlenk do not cover the Grassmannian as wrongly claimed in [21].

3.1 The work of Rudyak and Schlenk

The aim of the paper of Rudyak and Schlenk is to study the minimal number $S_B(M, \omega)$ of Darboux charts needed to parametrize a closed symplectic manifold (M, ω). Note that $S_B(M, \omega)$ is well defined due to Darboux theorem and the compactness of M. The estimate of $S_B(M, \omega)$ essentially consists of giving upper and lower bounds based on the Gromov width and the Lusternik-Schnirelmann category of M.
Let us explain how Rudyak and Schlenk used these two invariants to esti-

mate $S_B(M,\omega)$.
Denoting with $\mathcal{B}$ the image of a Darboux chart $\phi(B^{2n}) \subset M$ we can formally define:

$$S_B(M,\omega) := min\{k|M = \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_k\}$$

An immediate lower bound for the number $S_B(M,\omega)$ is given by the diffeomorphism invariant

$$\mathrm{B}(M) = min\{k|M = B_1 \cup \cdots \cup B_k\}$$

where each B_i is diffeomorphic to the open ball B^{2n}. This estimate from below depends only on the differential structure of M and thus it does not take in account the symplectic structure on the manifold.
We will now show another lower bound which depends on the symplectic structure. Consider the number

$$\gamma(M,\omega) := max\{Vol(B^{2n}(r))|B^{2n}(r) \text{ sympletically embeds in } M\}$$

It is clear that a bound from below for $S_B(M,\omega)$ is the integer

$$\Gamma(M,\omega) := \left\lfloor \frac{Vol(M,\omega)}{\gamma(M,\omega)} \right\rfloor + 1$$

where the braket $\lfloor x \rfloor$ denote the maximal integer smaller than or equal to x.

Remark. The number $\gamma(M,\omega)$ is equal to $\frac{1}{n!}(c_G(M,\omega))^n$ thus we can rewrite the last lower bound we introduced as

$$\Gamma(M,\omega) = \left\lfloor \frac{Vol(M,\omega)n!}{(c_G(M,\omega))^n} \right\rfloor + 1$$

Note also that the invariants $\Gamma(M,\omega)$ and $\mathrm{B}(M)$ have very different nature and are indeed not related in any way. The simplest example one can consider is the complex projective space $(\mathbb{C}P^n, \omega_{FS})$ for which we have $\Gamma(\mathbb{C}P^n, \omega_{FS})$=2 and $\mathrm{B}(\mathbb{C}P^n)$=$n$+1.
From this originates the need to set a lower bound which takes into account

both the differential and the symplectic structures on M:

$$\lambda(M,\omega) := max\{\Gamma(M,\omega); \mathrm{B}(M)\}$$

What we have shown so far is nothing more than

$$\lambda(M,\omega) \leq S_B(M,\omega) \tag{3.1}$$

We will focus on the invariant $\mathrm{B}(M)$ before we state the main result. We recall some invariants that give a estimate from below for $\mathrm{B}(M)$. The first is the *Lusternik* $-$ *Schnirelmann category* of M, that is defined for any CW-complex X:

$$\mathrm{Cat}(X) := min\{k|X = A_1 \cup \cdots \cup A_k\}$$

where each A_i is a open contractible subset of X. Obviously when M is a smooth closed manifold we have

$$\mathrm{Cat}(M) \leq \mathrm{B}(M).$$

In general the Lusternik-Schnirelmann category of a CW-complex X is not easy to compute, but may itself be estimated from below by the *cup lenght* of X which is defined as:

$$\mathrm{cl}(X) := sup\{k|u_1 \cdots u_k \neq 0, u_i \in \tilde{H}^*(X)\}$$

where $\tilde{H}^*(X)$ denotes the reduced singular cohomology of X with coefficients in any ring. It has been proved, for any CW-complex X, that

$$\mathrm{cl}(X) + 1 \leq \mathrm{Cat}(X)$$

and , for any smooth closed connected m-dimensional manifold M, that ([15]) $\mathrm{B}(M) \leq m+1$. Then, being M a smooth closed connected m-dimensional manifold, we can conclude that

$$\mathrm{cl}(M) + 1 \leq \mathrm{Cat}(M) \leq \mathrm{B}(M) \leq m+1. \tag{3.2}$$

This is not all we can say about these invariants, in fact in [23] is proved the following

Theorem 3.1 (Singhof)**.** *Let M^m be a close smooth p-connected manifold with $m \geq 4$ and* $\mathrm{Cat}(M) \geq 3$*. Then*

a) $\mathrm{B}(M) = \mathrm{Cat}(M)$ *if* $\mathrm{Cat}(M) \geq \dfrac{m+p+4}{2(p+1)}$,

b) $\mathrm{B}(M) \leq \left\lceil \dfrac{m+p+4}{2(p+1)} \right\rceil$ *if* $\mathrm{Cat}(M) < \dfrac{m+p+4}{2(p+1)}$.

Where $\lceil x \rceil$ denotes the minimal integer greater than or equal to x.

This result can be improved when we deal with symplectic manifolds:

Proposition 3.2. *Let (M, ω) be a $2n$-dimensional closed connected symplectic manifold. Then*

$$n+1 \leq \mathrm{cl}(M) + 1 \leq \mathrm{Cat}(M) \leq \mathrm{B}(M) \leq 2n+1. \tag{3.3}$$

Moreover the following claims hold true:

i) If $\pi_1(M) = 0$, then $n+1 = \mathrm{cl}(M)+1 = \mathrm{Cat}(M) = \mathrm{B}(M)$,

ii) If $[\omega]_{|\pi_2(M)} = 0$, then $\mathrm{Cat}(M) = \mathrm{B}(M) = 2n+1$,

iii) If $\mathrm{Cat}(M) < \mathrm{B}(M)$, then $n \geq 2$, $n+1 = \mathrm{cl}(M)+1 = \mathrm{Cat}(M)$ and $\mathrm{B}(M) = n+2$.

Proof. Since ω is a symplectic form we have $[\omega]^n \neq 0$ which implies $\mathrm{cl}(M) + 1 \geq n+1$. From this and from 3.2 we get 3.3. To prove assertions (i) to (iii) we will make use of Theorem 3.1. Note that we dropped the hypothesis $\dim(M) \geq 4$ and $\mathrm{Cat}(M) \geq 3$ in Theorem 3.1: in fact if $\dim(M) = 2$ we are in the case of closed orientable surfaces and it is easy to check that $\mathrm{B}(M) = \mathrm{Cat}(M)$; on the other hand if $\mathrm{Cat}(M) = 2$ then $\frac{1}{2}\dim(M) \leq \mathrm{cl}(M) + 1 \leq \mathrm{Cat}(M) = 2$ gives $\dim(M) = 2$.

i) If M is simply connected, it has been shown in [7] that $\mathrm{Cat}(M) \leq n+1$ and thus $\mathrm{Cat}(M) = n+1$. Since $p \geq 1$ we conclude that we are in the situation of Theorem 3.1 a) and it follows $\mathrm{Cat}(M) = \mathrm{B}(M) = n+1$.

ii) It has been proved that if $[\omega]_{|\pi_2(M)} = 0$ then $\mathrm{Cat}(M) = 2n+1$,(see [22]). From this and from $\mathrm{B}(M) \leq 2n+1$ follows the assertion.

iii) We know that $\mathrm{B}(M) = \mathrm{Cat}(M)$ if $n = 1$. Thus, assuming $\mathrm{B}(M) > \mathrm{Cat}(M)$, let $n \geq 2$. From i) we get $p = 0$ and the claim follows from Theorem 3.1 b).

□

We can now present the main result of [21]:

Theorem 3.3 (Rudyak-Schlenk). *Let (M, ω) be a closed connected symplectic manifold of dimension 2n.*

i) If $\lambda(M, \omega) \geq 2n+1$ then $S_B(M, \omega) = \lambda(M, \omega)$.

ii) If $\lambda(M, \omega) < 2n+1$ then $n+1 \leq \lambda(M, \omega) \leq S_B(M, \omega) \leq 2n+1$.

Idea of the proof: The proof of the theorem appears technical in several points, thus we will just present the idea behind the proof (for which Rudyak and Schlenk thank Gromov).
By inequalities 3.1 and 3.3 the assertion is a direct consequence of the following:

Theorem 3.4. *Let (M, ω) be a closed connected 2n-dimensional symplectic manifold.*

i) If $\Gamma(M, \omega) \geq 2n+1$, then $S_B(M, \omega) = \Gamma(M, \omega)$,

ii) If $\Gamma(M, \omega) \leq 2n+1$, then $S_B(M, \omega) \leq 2n+1$.

Let us denote with $\mu(A)$ the symplectic volume of any Borel set $A \subset M$ and set

$$k = max\{\Gamma(M, \omega); 2n+1\}.$$

Then by definition of $\Gamma(M, \omega)$ we have

$$\gamma(M, \omega) > \frac{\mu(M)}{k}.$$

Now by definition of $\gamma(M,\omega)$ there exists a Darboux chart $\varphi : B^{2n}(r) \to \mathcal{B} \subset M$ such that

$$\mu(\mathcal{B}) > \frac{\mu(M)}{k}.$$

From the last inequality and from $k \geq \dim(M)$, by means of elementary dimension theory, it is provided a covering of M consisting in k sets $\mathcal{C}^1 \ldots \mathcal{C}^k$ each of which is the image of a disjoint union of cubes in $\mathbb{R}^{2n}$ and such that

$$\mu(\mathcal{C}^j) < \mu(\mathcal{B}), \quad j = 1, \ldots, k.$$

It is then constructed a symplectomorphism $\Phi^j : M \to M$ such that $\Phi^j(\mathcal{C}^j) \subset \mathcal{B}$.

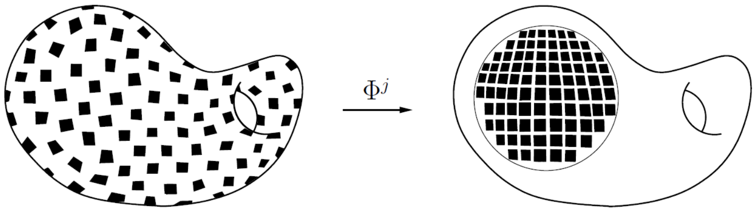


Figure 3.1: The idea behind the map Φ^j

It yields a symplectic cover of M made up of k charts:

$$(\phi^j)^{-1} \circ \varphi : B^{2n}(r) \to M,$$

and this ends the proof.
This Theorem basically reduces the issue of estimating $S_B(M,\omega)$ to two different problems: those of computing B(M) and $c_G(M,\omega)$. The results we know about B(M) are summarized in Proposition 3.2. Calculate $c_G(M)$ is instead an open and delicate problem in which there has recently been remarkable progress.
A related and more complicated problem than computing $S_B(M,\omega)$ is the one of symplectic packings. We say that a symplectic 2n-manifold (M,ω) admits a symplectic packing by N balls of radii $\lambda_1, \ldots, \lambda_N$ if there exist a

symplectic embedding from the disjoint union of the balls into (M, ω):

$$\varphi : \coprod_{j=1}^{N} \left(B^{2n}(\lambda_j), \omega_0\right) \to (M, \omega).$$

It is usually required a symplectic packing by N equal balls. Moreover we say that (M, ω) admits a full symplectic packing by N equal balls if the supremum of volumes which can be filled by symplectic embeddings of N disjoint equal balls is the volume of (M, ω) itself.
In analogy to this problem, Rudyak and Schlenk introduce the invariant $S_B^{=}(M, \omega)$ which is defined in the following way. First consider the number

$$S_B^r(M, \omega) := \min\{k | M = \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_k\}$$

where each $\mathcal{B}_j$ is the image of the same ball $B^{2n}(r)$ through a symplectic embedding. Now define

$$S_B^{=}(M, \omega) := \min_{r>0} S_B^r(M, \omega).$$

Remark. Theorem 3.3 holds true if we replace $S_B(M, \omega)$ with $S_B^{=}(M, \omega)$ since in the proof we constructed embeddings of equal balls.

Anyway, since we deal with $HSSCT$, we will not investigate further in this direction.

3.2 Some explicit computations

In this section we will present two examples which we find interesting in order to study the invariant $S_B(M, \omega)$ in the case of $HSSCT$.
We start with a very simple case

Riemann surfaces. Consider the case $n = 1$ that means we are dealing with closed oriented surfaces (Σ_g, ω) where ω is an area form. The following Lemma will give us the key ingredient.

Lemma 3.5 (Greene-Shiohama)**.** *Let U and V be two bounded domains in $(\mathbb{R}^2, \omega_0)$ which are diffeomorphic and have equal area. Then U and U are* *symplectomorphic.*

Proposition 3.6. *Let (Σ_g, ω) be a closed oriented surface of genus g. Then the following assertions hold true*

i) If $g = 0$, then $S_B(\Sigma_g, \omega) = 2$,

ii) If $g \geq 1$, then $S_B(\Sigma_g, \omega) = 3$.

Proof. From the previous Lemma we get $\mathrm{B}(\Sigma_g) = S_B(\Sigma_g, \omega)$. Now the claim follows from Proposition 3.2. □

Complex Projective spaces. Let $\mathbb{C}P^n$ be the n-dimensional complex projective space and let ω_{FS} the Fubini-Study form on $\mathbb{C}P^n$ normalized so that $\omega_{FS}(\mathbb{C}P^1) = \pi$.

Proposition 3.7. $S_B(\mathbb{C}P^n, \omega_{FS}) = n + 1$.

Proof. Since $\pi_1(\mathbb{C}P^n) = 0$ by Proposition 3.2.i) we get the inequality

$$S_B(\mathbb{C}P^n, \omega_{FS}) \geq \mathrm{B}(M) = n + 1.$$

On the other hand we can construct a symplectic atlas consisting in $n + 1$ charts. Consider for $0 \leq i \leq n$ the subsets of $\mathbb{C}P^n$

$$S_i = \{[z_0 : \ldots : z_n] \in \mathbb{C}P^n | z_i = 0\}$$

and the functions $f_i : B^{2n}(1) \to \mathbb{C}P^n$ defined by

$$f_i(\mathbf{z}) = f_i(z_1, \ldots, z_n) = [z_1 : \ldots : z_{i-1} : \sqrt{1 - |\mathbf{z}|^2} : z_{i+1} : \ldots, z_n] \tag{3.4}$$

which are well known to be symplectomorphisms between $(B^{2n}(1), \omega_0)$ and $(\mathbb{C}P^n \backslash S_i, \omega_{FS})$. Since $\mathbb{C}P^n = \bigcup_i f_i\big(B^{2n}(1)\big)$ we found an atlas $\{(B^{2n}(1), f_i)\}_{0 \leq i \leq n}$ that gives us the inequality

$$S_B(\mathbb{C}P^n, \omega_{FS}) \leq n + 1.$$

□

3.3 The case of the complex Grassmannian

The last result in [21] is the computation of $S_B(G_{k,n}, \omega_{FS})$ where $G_{k,n}$ is the complex Grassmannian of k-planes in $\mathbb{C}^n$ and ω_{FS} is, as usual, the Fubini-Study form on it. Note that this is a special case of our result presented in next chapter since $G_{k,n}$ is the compact dual of the bounded symmetric domain $I_{k,n-k}$. We will present the calculation of Rudyak and Schlenk which is based both on the knowledge of the volume and Gromov width of the Grassmannian manifold and on the construction of a specific symplectic atlas for $G_{k,n}$. We show next that the charts they claimed to make up the atlas do not cover the Grassmannian. Before that we need to introduce the notation.

We will consider only the case $n \geq 2k$ and $k \neq 1$ since $G_{k,n} = G_{n-k,n}$ and $G_{1,n} = \mathbb{C}P^{n-1}$. Let

$$P_{k,n} : G_{k,n} \to P(\wedge^k \mathbb{C}^n) = \mathbb{C}P^{\binom{n}{k}-1}$$

be the Plücker embedding and denote with $p_{k,n}$ its degree. It is well know that

$$p_{k,n} = \frac{(k-1)! \cdots 2!\, 1! \cdot (k(n-k))!}{(n-1)! \cdots (n-k+1)!\, (n-k)!}.$$

In order to underline the construction of Rudyak and Schlenk we split their result in two parts.

Proposition 3.8.

1. $S_B(G_{2,4}, \omega_{FS}) \in \{5, \dots, 9\}$,

2. $S_B(G_{2,5}, \omega_{FS}) \in \{7, \dots, 13\}$,

3. $S_B(G_{2,6}, \omega_{FS}) \in \{15, 16, 17\}$,

4. $S_B(G_{k,n}, \omega_{FS}) = p_{k,n} + 1 \quad$ *if* $\quad n \geq 7 \quad$ *or* $\quad k \geq 3$.

Proof. Since $G_{k,n}$is simply connected, from Proposition 3.2 follows

$$\mathrm{B}(G_{k,n}) = k(n-k) + 1.$$

Moreover

$$Vol(G_{k,n}, \omega_{FS}) = \frac{\pi^{k(n-k)} \cdot p_{k,n}}{(k(n-k))!}$$

Now, being $c_G(G_{k,n}, \omega_{FS}) = \pi$, a simple calculation yields

$$\Gamma(G_{k,n}, \omega_{FS}) = p_{k,n} + 1$$

and the claim follows directly from Theorem 3.3. $\square$

Remark. We will prove this Proposition in details in the more general context of $HSSCT$.

Proposition 3.9. *Embeddings in 3.4 can be generalized to $\binom{n}{k}$ embeddings $B^{2k(n-k)}(1) \to G_{k,n}$ covering $G_{k,n}$.*

Construction. The construction in [21] was first presented by Lu in [14], thus we refer to this last article.
Let us look at the matrix definition of the Grassmannian as the quotient

$$G_{k,n} = M(k,n)/GL(k)$$

where $M(k,n) = \{A \in \mathbb{C}^{k\times n} | \quad rank(A) = k\}$
and $GL(k) = \{Q \in \mathbb{C}^{k\times k} | \quad \det(Q) \neq 0\}$ acts on $M(k,n)$ from the left by matrix multiplication. Let $Pr : M(k,n)/GL(k) \to G_{k,n},\ A \mapsto [A]$ be the quotient projection and denote

$$M^0(k,n) = \{A \in M(k,n) | \quad A\overline{A}' = Id^k\}$$

where $\overline{A}'$ is the conjugate transpose of A and Id^k is the unit $k \times k$ matrix. The following Lemmata are the keys of the construction:

Lemma 3.10. *Let $\tau_0 = Pr^*(\omega_{FS})$, then $\tau_0|_{M^0(k,n)} = \omega_0|_{M^0(k,n)}$.*

Lemma 3.11. *The map*

$$\phi : (I_{k,n-k}, \omega_0) \to (\mathbb{C}^{k\times n}, \omega_0), \qquad Z \mapsto \left(\sqrt{Id^k - Z\overline{Z}'}, Z\right)$$

is a symplectic embedding with image in $M^0(k,n)$ and therefore it defines a symplectic embedding $\hat{\phi} = Pr \circ \phi$ of $(I_{k,n-k}, \omega_0)$ into $(G_{k,n}, \omega_{FS})$.

Now we need to use these result to construct $\binom{n}{k}$ embeddings of $B^{2k(n-k)}(1)$ into $G_{k,n}$.
In order to do this rewrite the matrix $A \in M(k,n)$ as $(A_1, \ldots, A_n)$ where A_j is the j-th column. Then for increasing integers $1 \leq \alpha_1 \leq \ldots \leq \alpha_k \leq n$ consider the complement $\{\alpha_{k+1}, \ldots, \alpha_n\}$ of $\{\alpha_1, \ldots, \alpha_k\}$ in the set $\{1, \ldots, n\}$. We write $A_{\alpha_1 \cdots \alpha_k}$ to indicate the matrix $(A_{\alpha_1}, \ldots, A_{\alpha_k})$.
Note that for every set $\{\alpha_1, \ldots, \alpha_k\}$ there exist a permutation matrix $P(\alpha_1, \ldots, \alpha_k)$ such that

$$(A_{\alpha_1}, \ldots, A_{\alpha_k}, A_{\alpha_{k+1}}, \ldots, A_{\alpha_n})P(\alpha_1, \ldots, \alpha_k) = (A_1, \ldots, A_n)$$

We can now define $\binom{n}{k}$ symplectic embeddings as

$$\begin{aligned} \hat{\phi}_{\alpha_1 \cdots \alpha_k} : (R_I^{k,n-k}, \omega_0) &\to (G_C, \omega_{k,n}) \\ Z &\mapsto \left[\left(\sqrt{Id^k - Z\overline{Z}'}, Z \right) P(\alpha_1, \ldots, \alpha_k) \right] \end{aligned}$$

Remark. The embedding $\hat{\phi}$ is exactly the embedding Ψ_M we constructed in the second chapter when $M = I_{k,n-k}$.

The claim of Rudyak and Schlenk is that the restrictions to $B^{2k(n-k)}(1)$ of the embeddings $\hat{\phi}_{\alpha_1 \cdots \alpha_k}$ cover $G_{k,n}$.

Proposition 3.12. *For every couple $\{k, n\}$ with $k \geq 2$ and $n \geq 2k$ there exist at least one point $[B] \in G_{k,n}$ such that for every set $\{\alpha_1 \cdots \alpha_k\}$ we have $[B] \notin \hat{\phi}_{\alpha_1 \cdots \alpha_k}\big(B^{2k(n-k)}(1)\big)$*

Proof. Let $A = \phi_{\alpha_1 \cdots \alpha_k}(Z)$ that means $\hat{\phi}_{\alpha_1 \cdots \alpha_k}(Z) = [A]$ and note that

$$A_{\alpha_1 \cdots \alpha_k} = \left(\sqrt{Id^k - Z\overline{Z}'} \right); \qquad A_{\alpha_{k+1} \cdots \alpha_n} = Z$$

We can easily compute $\|A\|^2$:

$$\begin{aligned} &\|A\|^2 = \|A_{\alpha_1 \cdots \alpha_k}\|^2 + \|A_{\alpha_{k+1} \cdots \alpha_n}\|^2 = \\ &tr(A_{\alpha_1 \cdots \alpha_k} \overline{A}'_{\alpha_1 \cdots \alpha_k}) + tr(A_{\alpha_{k+1} \cdots \alpha_n} \overline{A}'_{\alpha_{k+1} \cdots \alpha_n}) = \\ &tr(Id^k - Z\overline{Z}') + tr(Z\overline{Z}') = k \end{aligned}$$

From this calculation we get

$$\|A_{\alpha_1\cdots\alpha_k}\|^2 = k - \|A_{\alpha_{k+1}\cdots\alpha_n}\|^2 = k - \|Z\|^2$$

This shows that the image of $B^{2k(n-k)}(1)$ under $\hat{\phi}_{\alpha_1\cdots\alpha_k}$ is contained in

$$\Lambda^{k,n}_{\alpha_1\cdots\alpha_k} = \{[B] \in G_{k,n} | \forall A \in [B] \cap M^0(k,n), \|A_{\alpha_1\cdots\alpha_k}\|^2 > k-1\}$$

thus the set $\Lambda^{k,n}$ defined as

$$\{[B] \in G_{k,n} | \forall A \in [B] \cap M^0(k,n) \ \exists \{\alpha_1 \cdots \alpha_k\} \ s.t. \ \|A_{\alpha_1\cdots\alpha_k}\|^2 > k-1\}$$

contains the union of all the images $\hat{\phi}_{\alpha_1\cdots\alpha_k}(B^{2k(n-k)}(1))$.
Now, starting with the case $n = 2k$, we exhibit a point $[B] \notin \Lambda^{k,n}$. Consider then the $k \times k$ matrix C with entries $c_{i,i} = \frac{1}{\sqrt{2}}$ and $c_{i,j} = 0$ if $i \neq j$. Then the point $[B] = [(C, C)]$ does not belong to $\Lambda^{k,2k}$ since $B \in M^0(k, 2k)$ but on the other hand it is easy to see that

$$\max_{\{\alpha_1,\ldots,\alpha_k\}} \|B_{\alpha_1\cdots\alpha_k}\|^2 = \frac{k}{2} \leq k-1 \tag{3.5}$$

Now in case $n > 2k$ we set $[D] = [(C, C, 0, \ldots, 0)]$ where 0 denotes the column of zeroes. Clearly equation 3.5 holds if we replace B with D. $\square$

Remark. If we rewrite the results of Rudyak and Schlenk taking into account Proposition 3.12 we will find differences only in the cases of $G_{2,4}$, $G_{2,5}$ and $G_{2,6}$.

Note that proposition 3.12 does not imply that $G_{2,4}$, $G_{2,5}$ or $G_{2,6}$ do not admit a symplectic atlas of less then $\binom{n}{k}$ charts, on the contrary we believe that the following conjecture (which was stated in [21]) holds true.

Conjecture. $\lambda(M, \omega) = S_B(M, \omega)$ *for all closed connected symplectic manifold* (M, ω).

Chapter 4

Minimal symplectic atlases for HSSCT

In this final chapter we present our result about minimal atlases of $HSSCT$. In particular in the first section, combining the result of [12] and [21], we compute the invariant $S_B(M^*, \omega_{FS})$ where M^* is an irreducible $HSSCT$ of type $I_{k,n}, II_n$ or III_n. In the second part we focus on $HSSCT$ of type IV_n. Concretely, we provide a full symplectic embedding of the complex quadric.

4.1 Minimal symplectic atlases for $I_{k,n}, II_n$ and III_n

We focus now on the properties of $HSSCT$ which originate from their embedding in the complex projective spaces.
In general suppose that X is a $2n$-dimensional manifold which admits an holomorphic embedding $f : X \to \mathbb{C}P^d$. It is well know that X is a Kähler manifold when equipped with the form (which we denote as usual with ω_{FS}) induced by the Fubini-Study form on $\mathbb{C}P^d$. We can associate to f an integer, namely its degree $\deg(f)$, defined in the following way.
If $d > n$ there exist a point $p \in \mathbb{C}P^d$ such that $p \notin f(X)$. Up to composition with a unitary transformation of $\mathbb{C}P^d$ we can assume $p = [1 : 0 : \ldots : 0]$. Consider now the projection

$$p_d : \mathbb{C}P^d \to \mathbb{C}P^{d-1}; \quad [z_0 : \ldots : z_d] \mapsto [z_1 : \ldots : z_d]$$

and define the function $f_{d-1} = p_d \circ f$. Now iterating this argument we get a map $F : X \to \mathbb{C}P^n$ given by $F = p_{n+1} \circ \cdots \circ p_d \circ f$.
Then $\deg(f)$ is by definition $\deg(F)$ that is the integer such that

$$\int_X F^*\alpha = \deg(F) \int_{\mathbb{C}P^n} \alpha$$

where $[\alpha] \in H^{2n}(\mathbb{C}P^n, \mathbb{R})$. We can establish an important relation between the volume of X and that of $\mathbb{C}P^n$ via the degree of f.

Proposition 4.1. *Let X be a $2n$-dimensional manifold which admits an holomorphic embedding $f : X \to \mathbb{C}P^d$. Then*

$$\mathit{Vol}(X) = \deg(f)\, \mathit{Vol}(\mathbb{C}P^n)$$

Proof. Denote by $\omega_{FS}(d)$ (resp. $\omega_{FS}(n)$) the Fubini-Study form on $\mathbb{C}P^d$ (resp. $\mathbb{C}P^n$). Now consider the map $g = i \circ p_{n+1} \circ \cdots \circ p_d$ where i is the canonical inclusion map given by:

$$i : \mathbb{C}P^n \to \mathbb{C}P^d; \quad [z_0 : \ldots : z_n] \mapsto [0 : \ldots : 0 : z_{d-n} \ldots : z_d]$$

Keeping in mind that the function

$$\Phi : \mathbb{C}P^d \times [0,1] \to \mathbb{C}P^d; \;\; ([z_0 : \ldots : z_d], t) \mapsto [tz_1 : \ldots : tz_{d-n-1} : z_{d-n} \ldots : z_d]$$

is an homotopy between the identity map of $\mathbb{C}P^d$ and g we get

$$\begin{aligned} n!\mathrm{Vol}(X) = \int_X f^*\omega_{FS}(d)^n = \int_X (i \circ F)^*\omega_{FS}(d)^n = \\ \int_X F^*(i^*\omega_{FS}(d)^n) = \int_X F^*\omega_{FS}(n)^n = \\ \deg(F) \int_{\mathbb{C}P^n} \omega_{FS}(n)^n = \deg(f) n!\mathrm{Vol}(\mathbb{C}P^n). \end{aligned}$$

□

In consideration of Theorem 3.3 and the above result we deduce the following

Corollary 4.2. *Let (X, ω_{FS}) be a projectively induced Kähler manifold with $c_G(X, \omega_{FS}) = \pi r^2$. If $\dfrac{\deg(f)\pi^n}{c_G(X, \omega_{FS})^n} \geq 2n$ then $S_B(X, \omega_{FS}) = \deg(f)/r^{2n}+1$.*

Then we see that for projective induced Kähler manifold the computation of $S_B(X, \omega_{FS})$ is strictly related to the degree of the embedding $f : X \to \mathbb{C}P^d$. From now on (M^*, ω_{FS}) will be an irreducible $HSSCT$ with ω_{FS} normalized so that $\omega_{FS}(A) = \pi$ where $[A]$ is the generator of $H_2(M^*, \mathbb{Z})$ and $f : M^* \to \mathbb{C}P^d$ its holomorphic embedding. We compute now $S_B(M^*, \omega_{FS})$.
We have seen in section 2.3 that $c_G(M^*, \omega_{FS}) = \pi$. we can then rewrite Corollary 4.2 in the following form.

Corollary 4.3. *If (M^*, ω_{FS}) is an irreducible $HSSCT$ with* $\dim(M^*) = 2n$ *and* $\deg(f) \geq 2n$ *(or equivalently Vol$(M^*) \geq \pi^n \dfrac{2n}{n!}$) then $S_B(M^*, \omega_{FS}) =$* $\deg(f) + 1$

Note that an example of embedding $f : M^* \to \mathbb{C}P^d$ is given by the map σ in Theorem 2.5.
Now we need to know the degree of f or, equivalently, the volume of M^*. Recall that in section 2.2 we constructed a symplectic duality which, in particular, induces a symplectomorphism $\Phi : (\Omega, \omega_0) \to (M^* \setminus Y_p, \omega_{FS})$ where Ω is a bounded symmetric domain, M^* its compact dual and Y_p is the cut locus of a point $p \in M^*$. Being a symplectomorphism, Φ is also volume preserving. Thus we see that $\mathrm{Vol}(\Omega, \omega_0) = \mathrm{Vol}(M^*, \omega_{FS})$ since $\mathrm{Vol}(M^*, \omega_{FS}) = \mathrm{Vol}(M^* \setminus Y_p, \omega_{FS})$.
The knowledge of $\deg(f)$ is now reduced to that of $\mathrm{Vol}(\Omega, \omega_0)$. The volumes of classical irreducible bounded symmetric has been computed by Hua in [6] and we will refer to this. However a more general formula has been presented by Koranyi in [9] while Roos ([20]) proved that, when the volume element is suitably normalized, the volume of Ω equals the degree of f.
We report here the results of Hua for classical irreducible bounded symmetric

domains:

- $\mathrm{Vol}(I_{k,n-k}) = \dfrac{(k-1)!\cdots 2!1!}{(n-1)!\cdots(n-k+1)!(n-k)!}\pi^{k(n-k)}$

- $\mathrm{Vol}(II_n) = \dfrac{2!4!\cdots(2n-2)!}{n!(n+1)!\cdots(2n-1)!}\pi^{\frac{n(n+1)}{2}}$

- $\mathrm{Vol}(III_n) = \dfrac{2!4!\cdots(2n-4)!}{(n-1)!n!\cdots(2n-3)!}\pi^{\frac{n(n-1)}{2}}$

- $\mathrm{Vol}(IV_n) = \dfrac{2\pi^n}{n!}$

Note in particular that the result for $I_{k,n-k}$ is the same we used in the previous chapter. We are now ready to prove:

Theorem 4.4. *Let (M^*, ω_{FS}) be an irreducible compact Hermitian symmetric spaces of type I,II or III. If $\dim(M^*) = 2n$ is sufficiently large then Vol$(M^*) \geq \pi^n \dfrac{2n}{n!}$ and in particular $S_B(M^*, \omega_{FS}) = \deg(f) + 1$.*

Proof. This proof is actually a case by case verification. We denote by $f_{k,n}$ (resp. f_n) the embedding of the compact dual of an irreducible bounded symmetric domain of type $I_{k,n-k}$ (resp. II_n or III_n). Knowing $\deg(f)$ for all irreducible $HSSCT$ we show that, when the dimension is large enough, the relation $\deg(f) \geq \dim(M^*)$ holds and we can thus apply Corollary 4.3. We start with the irreducible $HSSCT$ of first type that is the complex Grassmannian $G_{k,n}$ of k-planes in $\mathbb{C}^n$. We need to show that

$$\frac{(k(n-k))! \cdot (k-1)!\cdots 2!1!}{(n-1)!\cdots(n-k+1)!(n-k)!} \geq 2k(n-k) \tag{4.1}$$

We will consider, as in previous chapter, $k \leq 2n$. A simple explicit calculation show that in the case $k = 2$ the first value of n such that equation (4.1) holds is $n = 7$. We show now that it holds also for each $n > 7$.
To do so we use a discrete version of ratio criterion that means we show that

$$\frac{\deg(f_{k,n+1})}{\deg(f_{k,n})} \geq \frac{2k(n-k+1)}{2k(n-k)}$$

Rewriting this explicitly we get

$$\frac{(2(n-1))!(n-2)!}{n!(2(n-2))!} \geq \frac{n-1}{n-2}$$

Thus reducing:

$$\frac{2(2n-3)}{n} \geq \frac{n-1}{n-2}$$

which is clearly true for $n \geq 7$.

Now consider the case $k \geq 3$. Note first first that equation (4.1) is satisfied when $(n, k) = (6, 3)$. We show now, using the ratio criterion, that it also holds for each couple $(2k, k)$, namely we prove that

$$\frac{\deg(f_{k+1,2k+2})}{\deg(f_{k,2k})} \geq \frac{2(k+1)^2}{2k^2}$$

Now again reducing we get the inequality:

$$\frac{k^2+1}{k+1} \cdots \frac{k^2+k}{2k} \cdot \frac{k^2+k+1}{k+1} \cdots \frac{k^2+2k+1}{2k+1} \geq \frac{(k+1)^2}{k^2}$$

which is true because each term in the left-hand member is greater than 3 while the right-hand one is always smaller than 2. With the same argument we can see that with fixed $k \geq 3$ we have

$$\frac{\deg(f_{k,n+1})}{\deg(f_{k,n})} \geq \frac{2k(n-k+1)}{2k(n-k)}$$

This ends the proof for the complex Grassmannian. In particular we can see that equation (4.1) is satisfied when $\dim(G_{k,n}) \geq 18$.

The proof for $HSSCT$ of type II_n and III_n follows exactly the same arguments we used above. For this reason we do not think it this useful to report it. This computation gives an explicit lower bound for the dimension of M^*. In particular if M^* is of type II_n or III_n then $\dim(M^*) \geq 30$.

□

Remark. We have proved a bit more than we have claimed. Indeed we have shown that the degree of f can be made arbitrary greater than the dimension of M^*.

Remark. The computation above is exactly the one we omitted in the proof of proposition 3.8.

Note that we cannot extend theorem 4.4 to the complex quadric Q_n since the condition $\deg(f) \geq \dim(Q_n)$ is never verified being $\deg(f) = 2$.

4.2 The case of Q_n

From the results of last section we deduce that the only case still open is that of the irreducible $HSSNT$ of type IV, namely the complex quadric $Q_n \subset \mathbb{C}P^{n+1}$. This section is then dedicate to investigate the symplectic geometry of Q_n.
In particular we provide here a full symplectic embedding of Q_n, i.e. a collection of symplectic embeddings $\varphi_i : B^{2n}(1) \to Q_n$ such that

$$\bigcup_i \overline{\varphi_i(B^{2n}(1))} = Q_n$$

This construction arises from the idea that, in view of Conjecture 3.3 one can provided a symplectic cover of Q_n consisting in $n+1$ charts.
We will find this full symplectic embedding using the theory we explained in section 2.2. Concretely we find the explicit form of the embedding $\Phi : B^{2n}(1) \to Q_n$ used in the computation of $c_G(M^*, \omega_{FS})$ and compose it with n translation giving rise to $n+1$ embeddings. We can resume the construction of Φ as follows:

$$B^{2n}(1) \subset (\mathbb{C}^n, h_0) \xrightarrow{\cong} (\mathcal{V}, (\cdot|\cdot)) \xrightarrow{i} (IV_n, \omega_0) \xrightarrow{\Psi_{IV_n}} (\mathcal{V}, \omega_{FS}) \xrightarrow{\xi} (Q_n, \omega_{FS})$$

where ξ is the Harish-Chandra embedding.
Let us first focus on the $HPJTS$ $\mathcal{V}$ associated to Q_n. The triple product on $\mathbb{C}^n$ making it a $HPJTS$ is given by

$$\{u, v, z\} = 2(h_0(u, \overline{v})z + h_0(z, \overline{v})u - h_0(u, \overline{z})v)$$

This is a simple $HPJTS$ of rank 2 with genus $g = n$. From the above definition we get

$$(u|v) = \frac{1}{n}\mathrm{tr}D(u, \overline{v}) = 2h_0(u, v)$$

which gives us the correspondence between $(\mathbb{C}^n, h_0)$ and $(\mathcal{V}, (\cdot|\cdot))$.
In order to have an explicit formula for Ψ_{IV_n} we need to understand the spectral decomposition of a regular point in $\mathcal{V}$. It is easy to verify that tripotents in $\mathcal{V}$ are the elements $c = x + iy$ (with $x, y \in \mathbb{R}^n$) such that

$$h_0(c, c) = \frac{1}{2}; \quad h_0(c, \overline{c}) = 0; \quad \|x\| = \|y\| = \frac{1}{2}$$

Now we want to find the spectral decomposition $v = \lambda_1 c_1 + \lambda_2 c_2$ of an element v. So let v be any element in $\mathcal{V}$ and define $\arg(v) = h_0(v, \overline{v})/\|h_0(v, \overline{v})\|$. For all $\mu \in \mathbb{C}$ such that $\|\mu\| = 1$ we have $\arg(\mu v) = \mu^2 \arg(v)$.
Thus if we set $\alpha = (\arg(v))^{\frac{1}{2}}$ then the element $v_+ = \overline{\alpha} v$ satisfies the equality $h_0(v_+, \overline{v}_+) = \|h_0(v, \overline{v})\|$. Now denote by x_+ (resp. y_+) the real (resp. imaginary) parts of v_+, that is $v_+ = x_+ + iy_+$.
It is not difficult to check that the spectral decomposition $v = \lambda_1 c_1 + \lambda_2 c_2$ is given by

$$\begin{aligned}
\lambda_1 &= \|x_+\| + \|y_+\| \\
c_1 &= \frac{\alpha}{2}\left(\frac{x_+}{\|x_+\|} + i\frac{y_+}{\|y_+\|}\right) \\
\lambda_2 &= \|x_+\| - \|y_+\| \\
c_1 &= \frac{\alpha}{2}\left(\frac{x_+}{\|x_+\|} - i\frac{y_+}{\|y_+\|}\right)
\end{aligned}$$

Moreover this decomposition satisfies the properties

$$h_0(v, v) = \frac{\lambda_1^2 + \lambda_2^2}{2}; \qquad \|h_0(v, \overline{v})\| = \lambda_1 \cdot \lambda_2$$

We can now understand the explicit form of the symplectic duality Ψ_{IV_n}.
If $v = \lambda_1 c_1 + \lambda_2 c_2$ is the spectral decomposition of $v \in \mathcal{V}$, from equation

(2.5) we get

$$B(v,v)c_i = (1-\lambda_i^2)^2 c_i$$

which in our case implies

$$z = \Psi_{IV_n}^{-1}(v) = \frac{\lambda_1}{(1+\lambda_1^2)^{1/2}} c_1 + \frac{\lambda_2}{(1+\lambda_2^2)^{1/2}} c_2 \tag{4.2}$$

Now $z \in B^{2n}(1) \subset IV_n$ if and only if $\|z\| < \frac{1}{2}$, i.e.

$$\left(\frac{\lambda_1}{(1+\lambda_1^2)^{1/2}}\right)^2 + \left(\frac{\lambda_2}{(1+\lambda_2^2)^{1/2}}\right)^2 < 1$$

Then a simple computation shows that if $z \in B^{2n}(1)$ then $v = \Psi_{IV_n}(z)$ must satisfy the condition

$$\|v'v\|^2 = \|h_0(v,\overline{v})\|^2 = (\lambda_1 \cdot \lambda_2)^2 < 1$$

Now in order to conclude the construction we need the expression of the Harish-Chandra map. Wolf computed (see [26]) the Harish-Chandra map $\xi : \mathcal{V} \to Q_n$ with basepoint $0 = [0 : \ldots : 0 : 1 : i] \in Q_n$. The explicit form of this map is:

$$\xi(v) = \xi(v_1, \ldots, v_n) = [2iv_1 : \ldots : 2iv_n : (1+v'v) : i(1-v'v)]$$

So far we have proved that

$$\Phi(B^{2n}(1)) = \{[z_0 : \ldots : z_{n+1}] \in Q_n | z_n \neq 0; z_{n+1} \neq 0\}$$

At this point we have a symplectic embedding of $B^{2n}(1)$ in Q_n, thus only need to construct n other embeddings such that the claim holds. Consider then the maps $P_i : \mathbb{C}^{n+2} \to \mathbb{C}^{n+2}$ for $i = 0, \ldots, n$ given by

$$f_i(z_0, \ldots, z_{n+1}) \mapsto (z_0, \ldots, z_{i-1}, z_n, z_{n+1}, z_i, \ldots, z_{n-1})$$

These are $n+1$ unitary isometries of $\mathbb{C}^{n+2}$, hence they induce holomorphic isometries F_i on Q_n. The embeddings we are looking for are then $\{F_i \circ \Phi\}$.

Now, as final step of our construction, we need to show that

$$\bigcup_i \overline{(F_i \circ \Phi)(B^{2n}(1))} = Q_n$$

Note that $(F_i \circ \Phi)(B^{2n}(1)) = \{[z_0 : \ldots : z_{n+1}] \in Q_n | z_i \neq 0; z_{i+1} \neq 0\}$.
Then $[z] = [z_0 : \ldots : z_{n+1}] \in Q_n$ is not in the image of $(F_i \circ \Phi)$ for $i = 0, \ldots, n$ if and only if there does not exist i such that $z_i \neq 0$ and $z_{i+1} \neq 0$.
Being $[z]$ a point of $\mathbb{C}P^{n+1}$ there exist $z_j \neq 0$. Let α be a complex number such that $\alpha z_j = 2$. Now from $z_{j+1} = 0$ we deduce

$$[\alpha z] \in \overline{(F_j \circ \Phi)(B^{2n}(1))}$$

We have then provided a full symplectic embedding of Q_n.
Unfortunately we cannot say anything about the invariant $S_B(Q_n, \omega_{FS})$ apart from what comes directly from Rudyak and Schlenk theorem that is

$$n + 1 \leq S_B(Q_n, \omega_{FS}) \leq 2n + 1$$

However we believe that it is possible to cover the complex projective quadric with $n + 1$ Darboux charts.

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