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SYMPLECTIC COORDINATES ON SYMMETRIC DOMAINS AND THEIR COMPACT DUAL

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SYMPLECTIC COORDINATES (1)

Let (M^{2n}, ω) be a symplectic manifold and let

$$\omega_0 = \sum_{j=1}^{n} dx_j \wedge dy_j$$

be the standard symplectic form on $\mathbb{R}^{2n}$.

Theorem (Darboux): For all $p \in M$ there exist an open set $U_p \subset M$ and a diffeomorphism

$$\psi_p : U_p \to \mathbb{R}^{2n}, \ \psi_p(p) = 0$$

such that $\psi_p^*(\omega_0) = \omega$.

Question: If $M \cong \mathbb{R}^{2n}$ can we take $U_p = M$?

Theorem (Gromov, Inv.Math. 1985): There exist exotic symplectic structures on $\mathbb{R}^{2n}$.

Corollary: the answer to the previous question is: NO.

SYMPLECTIC COORDINATES (2)

Theorem (McDuff, JDG 1988): Let (M, ω) be a Kähler manifold, $\pi_1(M) = \{1\}$. Assume that M is complete and $K \leq 0$. Then for all point $p \in M$ there exist a diffeomorphism

$$\psi_p : M \to \mathbb{R}^{2n}, \; \psi_p(p) = 0$$

satisfying $\psi_p^*(\omega_0) = \omega$.

Theorem (Ciriza, DGA 1993): Let $T \subset M$ be a complex and totally geodesic submanifold of M passing through p. Then, $\psi_p(T) = \mathbb{C}^k \subset \mathbb{C}^n, \dim_{\mathbb{C}} T = k$.

Question: What can we say when (M, ω) is an Hermitian symmetric space of noncompact type?

THE CASE OF THE DISK (1)

$\mathbb{C}H^1 = \{z \in \mathbb{C} \mid |z|^2 < 1\}$, $\omega = \omega_{hyp} = \frac{i}{2}\frac{dz\wedge d\bar{z}}{(1-|z|^2)^2}$

We look for a map

$$\psi : \mathbb{C}H^1 \to \mathbb{R}^2, \psi(0) = 0$$

such that

$$\psi^*(\omega_0) = \omega_{hyp},\ \omega_0 = dx \wedge dy$$

Assume $\psi(z) = f(r)z,\ r = |z|^2$.

Then $\psi^*(\omega_0) = \omega_{hyp}$ implies

$$\frac{\partial}{\partial r}(f^2 r) = \frac{1}{(1-r)^2} \Leftrightarrow f(r) = (1-r)^{-\frac{1}{2}}$$

Hence

$$\boxed{\psi(z) = \frac{z}{\sqrt{1-|z|^2}}}$$

THE CASE OF THE DISK (2)

Let $\mathbb{C}P^1$ be the one-dimensional complex projective space, (namely the compact dual of $\mathbb{C}H^1$) endowed with the Fubini–Study form ω_{FS}. Then we have the natural inclusions

$$\mathbb{R}^2 \cong \mathbb{C} \cong U_0 = \{z_0 \neq 0\} \subset \mathbb{C}P^1$$

Then

$$\omega_{FS}|_{U_0} = \frac{i}{2}\frac{dz \wedge d\bar{z}}{(1+|z|^2)^2}$$

and it is easily seen that

$$\psi^*(\omega_{FS}) = \omega_0,$$

where ω_0 is the restriction of ω_0 to $\mathbb{C}H^1 \subset \mathbb{C}$.

Summarizing we have proved a sort of "symplectic duality" between $(\mathbb{C}H^1, \omega_{hyp})$ and

$(\mathbb{C}P^1, \omega_{FS})$, namely there exists a diffeomorphism

$$\psi : \mathbb{C}H^1 \to \mathbb{R}^2 \cong \mathbb{C} \subset \mathbb{C}P^1$$

satisfying:

$$\boxed{\psi^*(\omega_0) = \omega_{hyp}}$$

$$\boxed{\psi^*(\omega_{FS}) = \omega_0}$$

BASIC EXAMPLE (1)

Let

$$D_I[n] = \{Z \in M_n(\mathbb{C}) \mid I_n - ZZ^* > 0\}$$

be the first Cartan domain equipped with the hyperbolic form

$$\omega_{hyp} = -\frac{i}{2}\partial\bar{\partial} \log \det(I_n - ZZ^*)$$

The compact dual of $D_I[n]$ is $\mathrm{Grass}_n(\mathbb{C}^{2n})$ endowed with the Fubini-Study form

$$\omega_{FS} = P^*(\omega_{FS}).$$

We have the following inclusions

$$D_I[n] \subset M_n(\mathbb{C}) = \mathbb{C}^{n^2} \subset \mathrm{Grass}_n(\mathbb{C}^{2n}) \stackrel{P=Plucker}{\hookrightarrow} \mathbb{C}P^N,$$

$N = \binom{2n}{n} - 1.$

BASIC EXAMPLE (2)

Theorem 1: The map

$$\Psi : D_I[n] \to M_n(\mathbb{C}) = \mathbb{C}^{n^2}$$

defined by

$$\Psi(Z) = (I_n - ZZ^*)^{-\frac{1}{2}} Z$$

is a diffeomorphism. Its inverse is given by

$$\Psi^{-1} : \mathbb{C}^{n^2} \to D_I[n],\ X \mapsto (I_n + XX^*)^{-\frac{1}{2}} X.$$

Moreover, Ψ is a *symplectic duality* namely,

$$\boxed{\Psi^*(\omega_0) = \omega_{hyp}}$$

$$\boxed{\Psi^*(\omega_{FS}) = \omega_0}$$

where

$$\boxed{\omega_0 = \tfrac{i}{2}\partial\bar{\partial}\,\mathrm{tr}(ZZ^*)}$$

Here

$$\omega_{FS} = \frac{i}{2}\partial\bar{\partial} \log\det(I_n + ZZ^*),\ \text{on } \mathbb{C}^{n^2} \subset \mathrm{Grass}_n(\mathbb{C}^{2n}).$$

BASIC EXAMPLE (3)

Proof of Theorem 1:

$$\omega_{hyp} = -\frac{i}{2}\partial\bar{\partial} \log \det(I_n - ZZ^*)$$

$$= \frac{i}{2} d\partial \log \det(I_n - ZZ^*) = \frac{i}{2} d\partial \, \mathrm{tr} \log(I_n - ZZ^*)$$

$$= \frac{i}{2} d \, \mathrm{tr} \, \partial \log(I_n - ZZ^*) = -\frac{i}{2} d \, \mathrm{tr}[Z^*(I_n - ZZ^*)^{-1} dZ]$$

By substituting $X = (I_n - ZZ^*)^{-\frac{1}{2}} Z$ one gets:

$$\omega_{hyp} = -\frac{i}{2} d \, \mathrm{tr}[Z^*(I_n - ZZ^*)^{-1} dZ]$$

$$= -\frac{i}{2} d \, \mathrm{tr}(X^* dX) + \frac{i}{2} d \, \mathrm{tr}\{X^* d[(I_n - ZZ^*)^{-\frac{1}{2}}] Z\}$$

Observe now that $-\frac{i}{2} d\,\mathrm{tr}(X^* dX) = \omega_0$ and

$$\mathrm{tr}[X^* d(I_n - ZZ^*)^{-\frac{1}{2}} Z] = d\,\mathrm{tr}(\frac{C^2}{2} - \log C),$$

where $C = (I_n - ZZ^*)^{-\frac{1}{2}}$.

JORDAN TRIPLE SYSTEMS

A **Hermitian Jordan triple system** is a pair $(\mathcal{M}, \{\,,\ ,\})$, where $\mathcal{M}$ is a complex vector space and $\{\,,\ ,\}$ is a $\mathbb{R}$-trilinear map

$$\{\,,\ ,\}: \mathcal{M}\times\mathcal{M}\times\mathcal{M} \to \mathcal{M}, (u,v,w) \mapsto \{u,v,w\}$$

$\mathbb{C}$-bilinear and simmetric in u and w and $\mathbb{C}$-antilinear in v and satisfying the **Jordan identity**:

$$\{x,y,\{u,v,w\}\} - \{u,v,\{x,y,w\}\} =$$

$$= \{\{x,y,u\},v,w\} - \{u,\{v,x,y\},w\}.$$

Let $u, v \in \mathcal{M}$, and let $D(u,v)$ be the operator on $\mathcal{M}$ defined by

$$\boxed{\text{D(u, v)(w)=\{u, v, w\}}}$$

A HJTS is called **positive** if

$$(u,v) \mapsto \operatorname{tr} D(u,v)$$

is positive definite.

A **HPJTS** is called *simple* if it is not the product of two non trivial sub-HPJTS.

The *quadratic representation*

$$Q : \mathcal{M} \to End(\mathcal{M})$$

is defined by

$$2Q(u)(v) = \{u,v,u\}, \; u,v \in \mathcal{M}.$$

The **Bergman operator**

$$B(u,v) : \mathcal{M} \to \mathcal{M}$$

is given by the equation

$$\boxed{B(u,v) = Id_{\mathcal{M}} - D(u,v) + Q(u)Q(v)}$$

HPJTS $\longrightarrow$ HSSNT

$(\mathcal{M}, \{,\ ,\}) \longrightarrow (M, 0) = \{u \in \mathcal{M} \mid B(u,u) >> 0\}_0$, where "$>>$" means positive definite w.r.t. $(u,v) \mapsto \operatorname{tr} D(u,v)$.

The **Bergman form** ω_{Berg} of M is defined as:

$$\omega_{Berg} = -\frac{i}{2}\partial\bar{\partial} \log \det B.$$

We also define (in the irreducible case)

$$\boxed{\omega_{hyp} = -\tfrac{i}{2}\partial\bar{\partial} \log \det B(z,z)}.$$

Remark: In general

$$\omega_{hyp} = -\frac{i}{2}\partial\bar{\partial} \log \mathcal{N}(z,z),$$

where $\mathcal{N}(z,z)$ is the so called *generic norm*.

If M is irreducible, or equivalently $\mathcal{M}$ is simple, $\det B = \mathcal{N}^g$.

HSSNT $\longrightarrow$ HPJTS

$(M, 0) \longrightarrow (\mathcal{M} = T_0 M, \{,\ ,\})$, where

$$\boxed{\{u, v, w\} = -\frac{1}{2}\left(R_0(u, v)w + JR_0(u, Jv)w\right)}$$

(see W. Bertam book LNM 1754 for the proof and related results)

THE BASIC EXAMPLE AS HPJTS

Let $\mathcal{M} = M_n(\mathbb{C})$ with the triple product

$$\{u, v, w\} = uv^*w + wv^*u, \;\; u, v, w \in M_n(\mathbb{C})$$

$$\operatorname{tr} D(u, u) = \operatorname{tr}(uu^*)$$

$$B(u, v)(w) = (I_n - uv^*)w(I_n - v^*u)$$

The HSSNT $(M, 0)$ associated to $(M_n(\mathbb{C}), \{, , \})$ is the first Cartan domain

$$D_I[n] = \{Z \in M_n(\mathbb{C}) \mid I_n - ZZ^* > 0\}$$

$$\omega_{hyp} = \frac{\omega_{Berg}}{2n} = -\frac{i}{2}\partial\bar{\partial} \log \det(I_n - ZZ^*)$$

COMPACTIFICATIONS OF HPJTS

Let (M, ω_{hyp}) be an HSSNT and let (M^*, ω_{FS}) be its compact dual equipped with the Fubini–Study form ω_{FS}.

More precisely, one has the following inclusions:

$$(M, 0) \overset{Harish-Chandra}{\subset} \mathcal{M} = T_0 M \overset{Borel}{\subset} M^* \overset{BW}{\hookrightarrow} \mathbb{C}P^N$$

and we set

$$\omega_{FS} = \mathrm{BW}^*(\omega_{FS}).$$

Remark: The local expression of ω_{FS} restricted to $\mathcal{M}$ is given (in the irreducible case) by

$$\boxed{\omega_{FS} = \frac{i}{2}\partial\bar{\partial} \log \det B(z, -z)}$$

Theorem 2 (Di Scala–Loi, 2006): The map

$$\boxed{\Psi_M : M \to \mathcal{M},\ z \mapsto B(z,z)^{-\frac{1}{4}}z}$$

satisfies the following properties:

(D) Ψ_M is a **diffeomorphism** and its inverse is given by

$$\boxed{\Psi_M^{-1} : \mathcal{M} \to M,\ z \mapsto B(z,-z)^{-\frac{1}{4}}z}$$

(H) the map

$$\Psi : HSSNT \to Diff_0(M, \mathcal{M}),\ M \mapsto \Psi_M$$

is **hereditary**, i.e.: for all $(T, 0) \stackrel{i}{\hookrightarrow} (M, 0)$ complete, complex and totally geodesic submanifold one has

$$\Psi_M|_T = \Psi_T,\ \Psi_M(T) = \mathcal{T} \subset \mathcal{M}$$

(S) Ψ_M is a **simplectic duality**, i.e.:

$$\boxed{\Psi_M^*(\omega_0) = \omega_{hyp}}$$

$$\boxed{\Psi_M^*(\omega_{FS}) = \omega_0}$$

where ω_0 is the flat Kähler form on $\mathcal{M}$.

Remark: In the irreducible case

$$\boxed{\omega_0 = \frac{i}{2g}\partial\bar{\partial}F},$$

where $F : \mathcal{M} \to \mathbb{R}, u \mapsto \mathrm{tr}\, D(u, u)$.

Remarks on Theorem 3

1. From the point of view of inducing geometric structures as in Gromov's programme the importance of property (S) relies on the existence of a smooth map which is a simultaneous symplectomorphism with respect to different symplectic structures.

2. Property (H) is exactly the above mentioned property observed by Ciriza for the McDuff map.

3. The map $\Psi_M : M \to \mathcal{M}$ above was defined, independently from the authors, by Guy Roos. He proved the analogous of (S) for volumes, namely

$$\Psi_M^*(\omega_0^n) = \omega_B^n,$$

$$\Psi_M^*((\omega_B^*)^n) = \omega_0^n$$

which follows from (S).

4. Guy Roos has pointed out that (D) and (S) of Theorem 3 can be proved by using the spectral decomposition theory of HPJTS.

Idea of the proof of (H) in Theorem 3

Proposition 3: Let $(M, 0)$ be a HSSNT and let $(\mathcal{M}, \{,\ ,\})$ be its associated HPJTS. Then there exists a bijection

$$\{(T, 0) \subset (M, 0)\} \longleftrightarrow \{\mathcal{T} \subset \mathcal{M}\},$$

where $\mathcal{T}$ is the HPJTS associated to T.

Property (H) follows by Proposition 3 combined with:

$$\boxed{\{u, v, w\} = -\frac{1}{2}\left(R_0(u, v)w + JR_0(u, Jv)w\right)}$$

Proofs of (D) e (S) for classical HSSNT

The proofs of **(D)** and **(S)** for classical $(C, 0) \in HSSNT$ are obtained combined (H) with the following

Proposition 4: Every classical HSSNT $(C, 0)$ admits a Kähler embedding into $(D_I[s], 0)$, for s sufficiently large.

JORDAN ALGEBRAS

A complex Jordan algebra is a complex vector space $\mathcal{A}$ endowed with a bilinear and symmetric product (non associative)

$$\circ : \mathcal{A} \times \mathcal{A} \to \mathcal{A}, \ (a, b) \mapsto a \circ b$$

such that:

$$a \circ (a^2 \circ b) = a^2 \circ (a \circ b), \forall a, b \in \mathcal{A},$$

dove $a^2 = a \circ a$.

Example:

$$\mathcal{A} = M_n(\mathbb{C}), \ u \circ v = \frac{uv + vu}{2}, \ u, v \in M_n(\mathbb{C}).$$

JORDAN ALGEBRAS AND HPJTS (1)

Let $(\mathcal{M}, \{\ ,\ ,\})$ be a HPJTS.

Assume that $\mathcal{M}$ admits a Jordan structure $\circ$ (i.e. $(\mathcal{M}, \circ)$ is a Jordan algebra) such that:

$$\{u, v, w\} = 2\left((u \circ \bar{v}) \circ w + (w \circ \bar{v}) \circ u - (u \circ w) \circ \bar{v}\right),$$

Then, the HSSNT $(M, 0)$ associated to $\mathcal{M}$ is called of *tube type*.

Example: $\{Z \in M_{m,n}(\mathbb{C}) \mid I_m - ZZ^* > 0\}$ is not of tube type.

Example: $D_I[n] = \{Z \in M_n(\mathbb{C}) \mid I_n - ZZ^* > 0\}$ is of tube type.

JORDAN ALGEBRAS AND HPJTS (2)

We have the following result

Lemma 5: Let $(M, 0)$ be a HSSNT and let $\mathcal{M}$ be its associated HPJTS. Then there exists a HSSNT $(\widetilde{M}, 0)$ such that:

(i) $(M, 0) \hookrightarrow (\widetilde{M}, 0)$ (complex and tot. geod.)

(ii) The HPJTS $\widetilde{\mathcal{M}}$ associated to $(\widetilde{M}, 0)$ arises from a Jordan algebra (equivalently $(\tilde{M}, 0)$ is of tube type).

Corollary 6: Let M be HSSNT, $p \in M$, $a, b \in T_pM$, $a, b \neq 0$ and let $\pi = \mathrm{span}_{\mathbb{C}}(a, b) \subset T_pM$. Then there exists a classical $C \hookrightarrow M$ passing through p such that $\pi \subset T_pC$.

Proof: Let assume that $p = 0 \in M$. Let $\mathcal{A}_{ab} \subset \widetilde{\mathcal{M}}$ be the Jordan subalgebra of $\widetilde{\mathcal{M}}$ spanned by a and b.

By a theorem of Jacobson-Shirsov the HSSNT associated to $\mathcal{A}_{ab}$ is of classical type. Thus by (i) of the previous lemma the HSSNT $C \hookrightarrow M \hookrightarrow \widetilde{M}$ associated to the HPJTS $\mathcal{A}_{ab} \cap \mathcal{M} \subset \mathcal{M}$ is as required.

Ideas of the proofs of (D) and (S) in the general case (1

1. First of all one has to prove that $\Psi_M^*(\omega_0)$ and $\Psi_M^*(\omega_{FS})$ are of type $(1,1)$.

2. Second, one can use Corollary 6 combined with the hereditary property (H) to reduce to the classical case (where we have already proved properties (D) and (S)).

Proof of $\Psi_M^*(\omega_0) = \omega_B$ under the assumption $\Psi_M^*(\omega_0)$ is of type $(1,1)$ (1)

Notice that

$$\omega_{\Psi_M} = \Psi_M^*(\omega_0) = \omega_B$$

is equivalent to

$$(\omega_{\Psi_M})_p(u, Ju) = (\omega_{hyp})_p(u, Ju),$$

$$(\omega_{\Psi_M})_p(Ju, Jv) = (\omega_{hyp})_p(Ju, Jv),$$

for all $p \in M$, $u, v \in T_pM$, where J denotes the almost complex structure of M evaluated at the point p. The second equation is precisely our assumption that $\Psi_M^*(\omega_0)$ is of type $(1,1)$.

Proof of $\Psi_M^*(\omega_0) = \omega_B$ under the assumption $\Psi_M^*(\omega_0)$ is of type $(1,1)$ (2)

Thus it remains to prove

$$(\omega_{\Psi_M})_p(u, Ju) = (\omega_{hyp})_p(u, Ju),$$

Fix $p \in M$ and $u \in T_pM$. Consider the complex line $\mathcal{L} = \mathrm{span}_{\mathbb{C}}(u) \subset T_pM$ and a classical complex and totally geodesic submanifold $(C, 0) \hookrightarrow (M, 0)$ such that $\mathcal{L} \subset T_pC$ (whose existence is guaranteed by Corollary 6). If we denote by $\omega_{hyp,C}$ and $\omega_{0,\mathcal{C}}$ the hyperbolic form on C and the flat Kähler form on $\mathcal{C}$ (the HPJTS associated to C) we get:

$$(\omega_{\Psi_M})_p(u, Ju) = (\Psi_C^*(\omega_{0,\mathcal{C}}))_p(u, Ju) =$$

$$= (\omega_{hyp,C})_p(u, Ju) = (\omega_{hyp})_p(u, Ju)$$

A result on the Bergman metric

As a byproduct of the previous proof one gets the following characterization of the Bergman metric on HSSNT.

Theorem: Let $(M, 0)$ be a HSSNT equipped with its Bergman form $\omega_{Berg,M}$. Let ω be a two form of type $(1,1)$ on M. Assume that the restriction of ω to all classical complex and totally geodesically submanifolds $(C, 0)$ (passing through the origin) equals the Bergman form of C. Then $\omega = \omega_{Berg,M}$.