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Balanced metrics, TYZ expansion and Szegö kernel

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BALANCED METRICS

Basic terminology

A *polarized manifold* (M, L) consists of a compact complex manifold M together with a very ample holomorphic line bundle $L \to M$.

Let (M, L) be a polarized manifold. A Kähler metric g on M such that $\omega_g \in c_1(L)$ is said to be *polarized by* L.

Let g be a Kähler metric on M polarized by L. Then there exists an hermitian metric h on L such that $Ric(h) = \omega_g$. Hence (L, h) is a *positive Hermitian line bundle* over M.

A *geometric quantization* of a Kähler manifold (M, ω_g) is a positive Hermitian line bundle (L, h) over M such that $Ric(h) = \omega_g$.

Kempf's distortion function and balanced metrics

Let (M, L) be a polarized manifold, g metric on M polarized by L and h Herm. metric on L such that $Ric(h) = \omega_g$.

Kempf's distortion function $T_g \in C^\infty(M, \mathbb{R}^+)$

$$T_g(x) = \sum_{j=0}^{N} h(s_j(x), s_j(x)), \ x \in M$$

where $\{s_0, \ldots, s_N\}$, $N + 1 = \dim H^0(L)$, is an o.b. with respect to:

$$\langle s, t \rangle_h = \int_M h(s, t) \frac{\omega_g^n}{n!}, \ s, t \in H^0(L)$$

Definition (Donaldson. JDG 2001): A polarized metric g on M is said to be *balanced* if $T_g = const = \frac{N+1}{V(M)}$, $V(M) = \int_M \frac{\omega_g^n}{n!}$.

Main results on balanced metrics

Theorem (G. Zhang, Comp. Math. '96): Let (M, L) be a polarized manifold. Then there exists a balanced metric g on M polarized by $L \Leftrightarrow (M, L)$ Chow polystable.

Theorem (Donaldson, JDG 2001): *Let (M, L) be a polarized manifold. Let g_{cscK} be a Kähler metric of constant scalar curvature polarized by L. Assume $\frac{\mathrm{Aut}(M,L)}{\mathbb{C}^*}$ discrete. Then, for all $m >> 1$, there exists a unique balanced metric g_m polarized by L^m and $\frac{g_m}{m} \xrightarrow{C^\infty} g_{cscK}$. Moreover, if g_m is a sequence of balanced metrics polarized by L^m such that $\frac{g_m}{m} \xrightarrow{C^\infty} g_\infty$ then g_∞ is cscK.*

Corollary: *Let* (M, L) *be a polarized manifold,* g_{cscK} *polarized by* L *and* $\frac{Aut(M,L)}{\mathbb{C}^*}$ *discrete. Then* (M, L) *is asymptotically Chow (poly)stable.*

Corollary: *Let* (M, L) *be a polarized manifold,* g_{cscK} *polarized by* L *and* $\frac{Aut(M,L)}{\mathbb{C}^*}$ *discrete. Then* g_{cscK} *is unique in* $c_1(L)$*.*

What happens without the assumption on **Aut**(M, L)

Theorem (C. Arezzo – L. , Comm. Math. Phys. 2004): *Let (M, L) be a polarized manifold and g and $\tilde{g}$ be two balanced metrics polarized by L. Then there exists $F \in$ Aut(M, L) such that $F^*\tilde{g} = g$.*

Theorem (A. Della Vedova – F. Zuddas, Trans. AMS, 2011): *Let $M = Bl_{p_1,\ldots,p_4}\mathbb{C}P^2$ (four points in the same line except one). Then there exists a polarization L of M and g_{cscK} polarized by L such that (M, L^m) is not Chow polystable for $m >> 1$.*

Theorem (X. Chen – G. Tian, Publ.Math.IHES, 2008): If $\omega_{\tilde{g}_{cscK}} \sim \omega_{g_{cscK}} \Rightarrow \exists\ F \in$ Aut(M) such that $F^*\tilde{g}_{cscK} = g_{cscK}$.

Some problems on balanced metrics

Let (M, L) be a polarized manifold.

$\mathcal{B}(L) = \{$balanced metrics on M polarized by $L^m,\ m = 1, \ldots\}$

$\mathcal{B}_c(L) = \{$equivalence classes of balanced metrics on $M\}$

where two balanced metrics in $\mathcal{B}(L)$ are equivalent iff they are polarized by L^{m_0} for some m_0.

$\mathcal{B}_{g_B} = \{m g_B \in \mathcal{B}(L) \mid m \in \mathbb{N}\}, \quad g_B \in \mathcal{B}(L)$

Problem: study $\#\mathcal{B}_c(L)$ and $\#\mathcal{B}_{g_B}$.

$$\#\mathcal{B}_{g_B} = \infty \implies \#\mathcal{B}_c(L) = \infty \ \underset{\Longrightarrow ?}{\Longleftarrow} \ (M, L) \text{ asynt.Chow pol.}$$

Balanced metrics and regular quantizations

Definition (M, Cahen, S. Gutt, J. Rawnsley, TRANS. AMS '83): Let (M, L) be a polarized manifold and g be a Kähler metric on M polarized by L. Then (L, h) is said to be a regular quantization of $(M, \omega_g = Ric(h))$ if mg is balanced $\forall m$.

$\#\mathcal{B}_{g_B} = \infty \Leftarrow (M, \omega_{g_B})$ reg. quant. $\Rightarrow (M, L)$ asynt. Chow pol.

$$\Uparrow$$

$$(M, g_{hom}),\ \pi_1(M) = 1,\ \omega_{g_{hom}} \text{ integral}$$

A conjecture and two theorems on balanced metrics

Conjecture: *Let (M, L) be a polarized manifold. If there exists $g_B \in \mathcal{B}(L)$ such that $\#\mathcal{B}_{g_B} = \infty$ then (M, g_B) is homogeneous and $\pi_1(M) = 1$.*

Theorem 1 (C. Arezzo, L. , F. Zuddas, Ann. Glob. Anal. Geom. 2011): *Let (M, L) be a polarized manifold. Assume* $\dim_{\mathbb{C}} M = 1$. *If there exists $g_B \in \mathcal{B}(L)$ such that $\#\mathcal{B}_{g_B} = \infty$ then $M = \mathbb{C}P^1$.*

Theorem 2 (C. Arezzo, L. , F. Zuddas, Ann. Glob. Anal. Geom. 2011): *Let M be a toric manifold,* $\dim M \leq 4$. *Let g_{KE} be a KE metric polarized by $L = K^*$. Then $\#\mathcal{B}_c(L) = \infty$. Moreover, there exists $g_B \in \mathcal{B}(L)$ such that $\#\mathcal{B}_{g_B} = \infty$ iff M is either the projective space or the product of projective spaces.*

TYZ (TIAN–YAU–ZELDITCH) EXPANSION

Balanced and projectively induced metrics

(M, L) polarized manifold, g polarized by L, $m \in \mathbb{N}^+$, h_m Hermitian metric on L^m such that $Ric(h_m) = m\omega_g$.

Let $\{s_0, \dots, s_{d_m}\}$, $d_m + 1 = \dim H^0(L^m)$, be an o.b. with respect to

$$\langle s, t\rangle_h = \int_M h_m(s,t)\frac{\omega_g^n}{n!}, s, t \in H^0(L^m),$$

$\varphi_m : M \to \mathbb{C}P^{d_m} : x \mapsto [s_0(x) : \dots : s_{d_m}(x)]$ *coherent states map*

$$\boxed{\varphi_m^*\omega_{FS} = m\omega_g + \frac{i}{2}\partial\bar{\partial}\log T_{mg}(x)}$$

$T_{mg}(x) = \sum_{j=0}^{d_m} h_m(s_j(x), s_j(x))$.

<u>Therefore:</u> mg is balanced $\Leftrightarrow$ mg is projectively induced by φ_m.

TYZ expansion

Theorem (S. Zelditch, Int. Math. Res. Not. '98): *Let* (M, L) *be a polarized manifold and* g *polarized by* L. *Then*

$$T_{mg}(x) \sim \sum_{j=0}^{\infty} a_j(x) m^{n-j}, \quad a_0(x) = 1,$$

namely, for all r *and* k *there exists* $C_{k,r}$ *such that*

$$||T_{mg}(x) - \sum_{j=0}^{k} a_j(x) m^{n-j}||_{C^r} \leq C_{k,r} m^{n-k-1}.$$

Corollary: *(Yau's conjecture proved by G. Tian JDG '90 in the* C^2 *case)* Let (M, L) be polarized manifold and g polarized by L. Then $\frac{\varphi_m^* g_{FS}}{m} \xrightarrow{C^\infty} g$.

On the coefficients of TYZ expansion

Theorem *(Z. Lu, Amer. J. Math. 2000):* Each $a_j(x)$ is a polynomial of the curvature of the metric g and of its covariant derivatives. Moreover,

$$\begin{cases} a_1(x) = \frac{1}{2}\rho \\ a_2(x) = \frac{1}{3}\Delta\rho + \frac{1}{24}(|R|^2 - 4|Ric|^2 + 3\rho^2) \\ a_3(x) = \frac{1}{8}\Delta\Delta\rho + \frac{1}{24}\,\mathrm{div}\,\mathrm{div}(R, Ric) - \frac{1}{6}\,\mathrm{div}\,\mathrm{div}(\rho Ric) + \\ +\frac{1}{48}\Delta(|R|^2 - 4|Ric|^2 + 8\rho^2) + \frac{1}{48}\rho(\rho^2 - 4|Ric|^2 + |R|^2) + \\ +\frac{1}{24}(\sigma_3(Ric) - Ric(R, R) - R(Ric, Ric)) \end{cases}$$

$$|R|^2 = \sum_{i,j,k,l=1}^n |R_{i\bar{j}k\bar{l}}|^2$$
$$|Ric|^2 = \sum_{i,j=1}^n |Ric_{i\bar{j}}|^2$$
$$|D^{'}\rho|^2 = \sum_{i=1}^n |\frac{\partial\rho}{\partial z_i}|^2$$
$$|D^{'}Ric|^2 = \sum_{i,j,k=1}^n |Ric_{i\bar{j},k}|^2$$
$$|D^{'}R|^2 = \sum_{i,j,k,l,p=1}^n |R_{i\bar{j}k\bar{l},p}|^2$$
$$div\, div(\rho Ric) = 2|D^{'}\rho|^2 + \sum_{i,j=1}^n Ric_{i\bar{j}}\frac{\partial^2\rho}{\partial\bar{z}_j\partial z_i} + \rho\Delta\rho$$
$$div\, div(R, Ric) = -\sum_{i,j=1}^n Ric_{i\bar{j}}\frac{\partial^2\rho}{\partial\bar{z}_j\partial z_i} - 2|D^{'}Ric|^2$$
$$+\sum_{i,j,k,l=1}^n R_{j\bar{i}l\bar{k}}R_{i\bar{j},k\bar{l}} - R(Ric, Ric) - \sigma_3(Ric)$$
$$R(Ric, Ric) = \sum_{i,j,k,l=1}^n R_{i\bar{j}k\bar{l}}Ric_{j\bar{i}}Ric_{l\bar{k}}$$
$$Ric(R, R) = \sum_{i,j,k,l,p,q=1}^n Ric_{i\bar{j}}R_{j\bar{k}p\bar{q}}R_{k\bar{i}q\bar{p}}$$
$$\sigma_3(Ric) = \sum_{i,j,k=1}^n Ric_{i\bar{j}}Ric_{j\bar{k}}Ric_{k\bar{i}},$$

where " ,p" is the covariant derivative in the direction $\frac{\partial}{\partial z_p}$*.*

The proof Theorem 1 and 2

Lemma 1: *Let* (M, L) *be a polarized manifold and* g *polarized by* L. *Let* $\mathcal{B}_g = \{mg \text{ is balanced} \mid m \in \mathbb{N}\}$. *If* $\#\mathcal{B}_g = \infty$ *then the coefficients* $a_j(x)$ *of* $T_{mg}(x) \sim \sum_{j=0}^{\infty} a_j(x) m^{n-j}$ *are constants for all* $j = 0, 1, \ldots$.

proof: *Let* $\{m_s\}_{s=1,2,\ldots}$ *be an unbounded sequence such that* $T_{m_s g}(x) = T_{m_s}$. *We know that* $a_0 = 1$. *Assume that* $a_j(x) = a_j$, *for* $j = 0, \ldots, k-1$. *Then,*

$$|T_{s,k,n} - a_k(x) m_s^{n-k}| \leq C_k m_s^{n-k-1}, \quad T_{s,k,n} = T_{m_s} - \sum_{j=0}^{k-1} a_j m_s^{n-j}$$

for some constants C_k.

Then $|m_s^{k-n} T_{s,k,n} - a_k(x)| \leq C_k m_s^{-1}$ *and if* $s \to \infty$ *then* $m_s^{k-n} T_{s,k,n} \to a_k(x)$ *and hence* a_k *is costant.* $\square$

The proof of Theorem 1

Theorem 1 *(C. Arezzo, L. , F. Zuddas, Ann. Glob. Anal. Geom. 2011):* Let (M, L) be a polarized manifold. Assume $\dim_{\mathbb{C}} M = 1$. If there exists $g_B \in \mathcal{B}(L)$ such that $\#\mathcal{B}_{g_B} = \infty$ then $M = \mathbb{C}P^1$.

proof:

If $\#\mathcal{B}_{g_B} = \infty \stackrel{Lemma 1}{\Longrightarrow} a_j^B$ $(T_{mg_B}(x) \sim \sum_{j=0}^{\infty} a_j^B(x) m^{n-j})$ *are constants for all* $j = 0, 1, \ldots$

In particular $a_1^B = \rho_B/2$ *is constant* $\stackrel{Calabi, Ann. Math. `53}{\Longrightarrow}$ $M = \mathbb{C}P^1$ *and* $g_B = m_0 g_{FS}$. $\square$

Lemma 2: Let (M, L) be a polarized manifold and $g = g_{cscK}$ polarized by L. Assume that mg is not proj. induced $\forall m$. Then $\#\mathcal{B}_{g_B} < \infty$ for all $g_B \in \mathcal{B}(L)$.

proof: *Let* $g_B \in \mathcal{B}(L)$ *(*g_B *balanced and* $g_B \in c_1(L^{m_0})$ *for some* m_0*).*

If $\#\mathcal{B}_{g_B} = \infty \overset{\text{Lemma 1}}{\Longrightarrow} a_j^B$ $(T_{mg_B}(x) \sim \sum_{j=0}^{\infty} a_j^B(x) m^{n-j})$ *are constants for all* $j = 0, 1, \ldots$.

In particular $a_1^B = \rho_B/2$ *is constant and hence (by Chen–Tian theorem) there exists* $F \in$ *Aut*(M) *such that* $F^* g_B = m_0 g$.

This implies that $m_0 g$ *is proj. induced in contrast with the assumptions.* $\square$

Remark: *There exist* g_{cscK} *polarized by* L *such that all the coefficients of TYZ are costants but* mg *is not projectively induced for all* m *(e.g. hyperbolic metrics, flat metrics on abelian varieties).*

Sketch of the proof of Theorem 2

Theorem 2 *(C. Arezzo, L. , F. Zuddas, Ann. Glob. Anal. Geom. 2011):* Let M be a toric manifold, $\dim M \leq 4$. Let g_{KE} be a KE metric polarized by $L = K^*$. Then $\#\mathcal{B}_c(L) = \infty$. Moreover, there exists $g_B \in \mathcal{B}(L)$ such that $\#\mathcal{B}_{g_B} = \infty$ iff M is either the projective space or the product of projective spaces.

idea of the proof:

$\#\mathcal{B}_c(L) = \infty$ *follows by the fact that symmetric toric manifolds* $(M, L = K^*)$ *are asympt. Chow polystable.*

Hard part: $m_0 g_{KE}$ *is proj. induced for some* m_0 *iff* M *is either the projective space or the product of projective spaces. Conclusion follows by Lemma 2.* $\square$

Conjecture: Every KE submanifold of $\mathbb{C}P^N$ is homogeneous.

Remark: There exist non homogeneous and complete KE submanifolds of $\mathbb{C}P^\infty$ (L., M. Zedda, Math. Ann. 2011)

SZEGÖ KERNEL

The unit disk bundle and the circle bundle in L^*

Let (L, h) *be a positive Hermitian line bundle over a compact Kähler manifold* (M, g) *of complex dimension* n*, such that* $Ric(h) = \omega_g$*. Consider the negative Hermitian line bundle* (L^*, h^*) *over* (M, g) *dual to* (L, h)*.*

Let $D \subset L^*$ *be the unit disk bundle over* M*, i.e.*

$$\boxed{D = \{v \in L^* \mid \rho(v) = 1 - h^*(v, v) > 0\}}$$

The condition $Ric(h) = \omega_g$ *implies that* D *is strongly pseudoconvex domain in* L^* *with smooth boundary (Grauert, '50).*

Let $\boxed{X = \partial D = \{v \in L^* \mid \rho(v) = 0\}}$ *be the unit circle bundle*

The Szegö kernel of the disk bundle

Consider the separable Hilbert space $\mathcal{H}^2(D)$ *consisting of holomorphic functions* $f : D \to \mathbb{C}$, $f \in C^0(\bar{D})$, *such that*

$$\int_X |f|^2 d\mu < \infty, \quad d\mu = \alpha \wedge (d\alpha)^n, \ \alpha = -i\partial\rho_{|X} = i\bar{\partial}\rho_{|X}$$

Let $\{f_j\}_{j=1,\dots}$ *be an orthonormal basis of* $\mathcal{H}^2(D)$, *i.e.*

$$\int_X f_j \bar{f}_k d\mu = \delta_{jk}.$$

The Szegö kernel is defined by:

$$\mathcal{S}(v) = \sum_{j=1}^{+\infty} f_j(v)\overline{f_j(v)}, \ v \in D.$$

Szegö kernel and Kempf's distortion function

Theorem *(S. Zelditch, Int. Math. Res. Not. '98)*

$$\mathcal{H}^2(D) = \oplus_{m=0}^{+\infty} \mathcal{H}^2_m(D)$$

$$\mathcal{H}^2_m(D) = \{f \in \mathcal{H}^2(D) \mid f(\lambda v) = \lambda^m f(v),\ \lambda \in S^1\}$$

The map $s \in H^0(L^m) \mapsto \widehat{s} \in \mathcal{H}^2_m(D)$ given by:

$$\widehat{s}(v) = v^{\otimes m}(s(x)),\ x = \pi(v),\ \pi : L^* \to M.$$

is an isometry between $H^0(L^m)$ and $\mathcal{H}^2_m(D)$. Moreover:

$$\boxed{\mathcal{S}_m(v) = \sum_{j=0}^{d_m} \widehat{s}_j(v)\overline{\widehat{s}_j(v)} = \sum_{j=0}^{d_m} h_m(s_j(x), s_j(x)) = T_{mg}(x),\ v \in X}$$

The log term of the Szegö kernel

Theorem *(C. Fefferman, BULL. AMS '83):* There exist $a, b \in C^\infty(\bar{D})$, $a \neq 0$ on $X = \partial D$ such that:

$$\boxed{\mathcal{S}(v) = a(v)\rho(v)^{-n-1} + b(v)\log\rho(v), \;\; v \in D}$$

where $\rho(v) = 1 - h^*(v, v)$ is the defining function of D.

Definition: *One says that the log term of the Szegö kernel of the disk bundle $D \subset L^*$ vanishes if $b = 0$.*

On the vanishing of the log term of the Szegö kernel

Theorem (G. Tian – Z Lu, Duke 2004): *Let* (M, L) *be a polarized manifold and* g *be a Kähler metric on* M *polarized by* L*. Let* h *be an Hermitian product on* L *such that* $\omega_g = Ric(h)$*. If the log term of the Szegö kernel of* $D = \{v \in L^* \mid h^*(v, v) < 1\}$ *vanishes then* $a_k = 0$ *for* $k > n$*.* ($T_{mg}(x) \sim \sum_{j=0}^{\infty} a_j(x) m^{n-j}$)

The case of $\mathbb{C}P^n$

Example: $(L = O(1), h_{FS}) \to (\mathbb{C}P^n, \omega_{FS})$, $Ric(h_{FS}) = \omega_{FS}$,

$D = \{v \in L^* = O(-1) \mid h^*_{FS}(v, v) < 1\}$

$X = \partial D = S^{2n+1} \to \mathbb{C}P^n$ Hopf fibration.

One can prove that the log term of the Szegö kernel of D vanishes.

A conjecture for $\mathbb{C}P^n$

Conjecture: (G. Tian – Z. Lu, 2004): *Let h be an Hermitian metric on $L = O(1) \to \mathbb{C}P^n$ such that $Ric(h) = \omega \sim \omega_{FS}$. Assume that the log term of the Szegö kernel of $D = \{v \in L^* = O(-1) \mid h^*(v,v) < 1\}$ vanishes then there exists $F \in$ Aut$(\mathbb{C}P^n)$ such that $F^*\omega = \omega_{FS}$.*

The conjecture of Lu and Tian holds true for $\mathbb{C}P^1$

Theorem (G. Tian – Z. Lu, Duke 2004): Let h be an Hermitian metric on $L = O(1) \to \mathbb{C}P^1$ such that $Ric(h) = \omega \sim \omega_{FS}$. Assume that the log term of the Szegö kernel of

$D = \{v \in L^* = O(-1) \mid h^*(v, v) < 1\}$ vanishes

then there exists $F \in \mathrm{Aut}(\mathbb{C}P^1)$ such that $F^*\omega = \omega_{FS}$.

proof:

$$a_2(x) = \frac{1}{3}\Delta\rho + \frac{1}{24}(|R|^2 - 4|Ric|^2 + 3\rho^2) = \frac{1}{3}\Delta\rho = 0 \Rightarrow \rho = const. \square$$

The conjecture of Lu and Tian holds true locally

Theorem (G. Tian – Z. Lu, Duke 2004): *There exists* $\epsilon = \epsilon(n)$ *such that if* h *is an Hermitian metric on* $L = O(1) \to \mathbb{C}P^n$ *such that:*

1. $\|\frac{h}{h_{FS}} - 1\|_{C^{2n+4}} < \epsilon$;

2. *the log term of the Szegö kernel of*

$$D = \{v \in L^* = O(-1) \mid h^*(v, v) < 1\}$$

vanishes;

then there exists $F \in$ *Aut*$(\mathbb{C}P^n)$ *such that* $F^*\omega = \omega_{FS}$, $\omega = Ric(h)$.

Theorem 3 (D. Uccheddu, 2011) *Let* $M = \mathbb{C}P^2$ *and* $\omega_\alpha = f_\alpha^* \omega_{FS}$ *be the Kähler form on* $\mathbb{C}P^2$ *obtained as the pull-back of* ω_{FS} *on* $\mathbb{C}P^5$ *via the map:*

$f_\alpha : \mathbb{C}P^2 \to \mathbb{C}P^5 : [Z_0, Z_1, Z_2] \mapsto [Z_0^2, Z_1^2, Z_2^2, \alpha Z_0 Z_1, \alpha Z_0 Z_2, \alpha Z_1 Z_2].$

Let h_α be the Hermitian metric on $O(2)$ such that

$$Ric(h_\alpha) = \omega_\alpha \sim 2\omega_{FS}.$$

Assume that the log term of the disk bundle

$$D_\alpha = \{v \in O(2) \mid h_\alpha(v, v) < 1\}$$

vanishes. Then $|\alpha|^2 = 2$, i.e. $\omega_\alpha = 2\omega_{FS}$.

Some problems on the Szegö kernel of the disk bundle

1. Classify the Kähler manifolds where $a_k = 0$, for $k > n$. Is it true that the Szegö kernel of the disk bundle $D \subset L^*$ associated to such manifolds has vanishing log term?

Remark: *For all $k \geq 1$ the equation (for ω and f fixed) $a_k(\omega + \frac{i}{2}\partial\bar{\partial}\varphi) = f$ is an elliptic PDE (Tian – Lu, 2004).*

2. Find examples of Kähler manifolds different from the projective space whose Szegö kernel of the disk bundle $D \subset L^*$ has vanishing log term.

3. Is it true that the vanishing of the log term of the disk bundle $D \subset L^*$ implies some topological restrictions on $X = \partial D$, e.g. X is homeomorphic to S^{2n+1}?

Ramadanov's conjecture

Conjecture: *(I.P. Ramadanov, C. R. Acad.Bulgare Sci.1981)* Let D be a bounded strongly pseudoconvex domain of $\mathbb{C}^n$ with smooth boundary. If the log term of the Bergman kernel of D vanishes then D is biholomorphic to the ball.

The work of M. Engliš on problems 2 and 3

Theorem: *(M. Engliš, G. Zhang, Math.Z. 2010) Let* (M, g) *be a Hermitian symmetric space of compact type of complex dimension* n*. Assume* ω_g *integral and let* (L, h) *be the Hermitian line bundle such that* $Ric(h) = \omega_g$*. Then the log terrn of the Szegö kernel of the disk bundle* $D \subset L^*$ *vanishes. Moreover* $X = \partial D$ *is homeomorphic to* S^{2n+1} *iff* $M = \mathbb{C}P^n$*.*

Regular quantizations and Szegö kernel

Theorem 4: (C. Arezzo, L., F. Zuddas, 2011) *Let g be a Kähler metric on M polarized by L. If (L, h) is a regular quantization of (M, ω_g), $Ric(h) = \omega_g$, then the log term of the disk bundle $D = \{v \in L^* \mid h^*(v, v) < 1\}$ vanishes.*

Corollary: Let (M, g) be a homogeneous compact and simply-connected Kähler manifold of complex dimension n. Assume ω_g integral and let (L, h) be the Hermitian line bundle such that $Ric(h) = \omega_g$. Then the log terrn of the Szegö kernel of the disk bundle $D \subset L^*$ vanishes. Moreover $X = \partial D$ is homeomorphic to S^{2n+1} iff $M = \mathbb{C}P^n$.