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Kähler immersions of homogeneous Kähler manifolds into complex space forms

joint with

Antonio J. Di Scala (Politecnico di Torino)

and

Hideyuki Hishi (Nagoya University)

Aim. *Classify all homogeneous Kähler manifolds which admit a Kähler immersion into a given finite or infinite dimensional complex space form.*

Kähler manifolds

A *Kähler manifold* (M, g) is a complex manifold $M = (M, J)$ equipped with a Riemmannian metric g such that the two-form ω on M defined by

$\omega(X, Y) \stackrel{\text{def}}{=} g(X, JY)$, $X, Y \in \mathfrak{X}(M)$, is closed, i.e. $d\omega = 0$.

The form ω is called the *Kähler form associated to the metric* g.

On a contractible open set $U \subset M$

$$\omega = \frac{i}{2}\partial\bar{\partial}\Phi = \frac{i}{2}\sum_{j=1}^{n} \frac{\partial^2\Phi}{\partial z_j \partial \bar{z}_k} dz_j \wedge d\bar{z}_k,$$

where $\Phi : U \to \mathbb{R}$ is a strictly PSH function called a *Kähler potential* for the metric g.

Complex space forms

A *complex space form* $(S, g_S) = (S, g_S, \omega_S, J_S)$ is a finite or infinite dimensional Kähler manifold of constant holomorphic sectional curvature.

Classification of complex space forms

Complex Euclidean space $(\mathbb{C}^N, g_0)$, $\mathbb{C}^\infty \stackrel{def}{=} \ell^2(\mathbb{C})$ $(z = \{z_j\} \in \ell^2(\mathbb{C})$ iff $\sum_{j=1}^{\infty} |z_j|^2 < \infty)$

$\omega_0 = \frac{i}{2}\partial\bar{\partial}|z|^2 = \sum_{j=1}^{N} dz_j \wedge d\bar{z}_j$, $|z|^2 = |z_1|^2 + \cdots + |z_N|^2$.

Complex hyperbolic space $(\mathbb{C}H^N = \{z \in \mathbb{C}^N \mid |z|^2 < 1\}, g_{hyp})$,

$\omega_{hyp} = -\frac{i}{2}\partial\bar{\partial}\log(1 - |z|^2)$.

Complex projective space $(\mathbb{C}P^N = \mathbb{C}^{N+1} \setminus \{0\}/z \sim \lambda z, g_{FS})$. In the chart $U_0 = \{Z_0 \neq 0\}$

$\omega_{FS} = \frac{i}{2}\partial\bar{\partial}\log(1 + |z|^2)$, $z_j = \frac{Z_j}{Z_0}$, $j = 1, \ldots, N$.

Kähler immersions into complex space forms

Let (M, g) be a Kähler manifold. A Kähler immersion $f : M \to (S, g_S)$ is a holomorphic map (i.e. $df \circ J = J_S \circ df$) which is isometric (i.e. $f^* g_S = g$).

Remark The word *immersion* is redundant.

Remark A Kähler immersion $f : (M, g) \to (S, g_S)$ is symplectic, namely $f^* \omega_S = \omega$. Viceversa a holomorphic and symplectic map $f : M \to S$ is isometric, i.e. $f^* g_S = g$.

Calabi's results on Kähler immersions (Ann. Math. 1953)

Theorem. (Calabi's rigidity) *If $f : (M, g) \rightarrow (S, g_S)$ is a Kähler immersion then any other Kähler immersion of (M, g) into (S, g_S) is given by $\mathcal{U} \circ f$ where $\mathcal{U}$ is a unitary transformation, i.e. $\mathcal{U} \in$ Aut$(S) \cap$ Isom(S, g_S).*

Theorem. (local immersions vs global immersions) *A simply-connected real-analytic Kähler manifold (M, g) admits a Kähler immersion into a given complex space form (S, g_S) iff there exists an open set $U \subset M$ such that $(U, g_{|U})$ can be Kähler immersed into (S, g_S).*

Homogeneous Kähler manifolds: definitions

A homogeneous Kähler manifold (h.K.m.) is a Kähler manifold (M, g) *such that the Lie group* $G = \mathit{Aut}(M) \cap \mathit{Isom}(M, g)$ *acts transitively on* M*.*

Remark. The metric g is not uniquely determined by G. There exist different (neither homothetic or isometric) G-invariant homogeneous metrics.

Homogeneous bounded domains

Let $\Omega \subset \mathbb{C}^n$, Ω bounded domain endowed with a homogeneous Kähler metric g_Ω. Then (Ω, g_Ω) is called a *homogeneous bounded domain* (h.b.d.).

If Aut(Ω) acts transitively on $\Omega \subset \mathbb{C}^n$ then (Ω, g_B) is a h.b.d..

Remark. Every bounded symmetric domain (Ω, g_B) (where the geodesic symmetry $\exp_x(v) \mapsto \exp_x(-v), \forall x \in M, v \in T_xM$ is holomorphic and an isometry) is a h.b.d. but there exist (Pyatetskii-Shapiro, 1969) h.b.d. (Ω, g_B) which are not bounded symmetric domains.

Other examples of h.K.m.

Flat h.K.m. $\mathcal{E} = \mathbb{C}^n \times T_1 \times \cdots T_k$ where $T_j = \mathbb{C}^{n_j}/\Lambda_j$ is a complex torus with the flat metric.

Compact simply-connected h.K.m. These are also called *Kähler C-spaces* or *Wang's spaces* or *rational homogeneous varieties*.

Compact h.K.m. $(M, g) = \mathcal{C} \times T_1 \times \cdots T_k$, C-space, T_j flat torus.

Products of homogeneous Kähler manifolds The products of h.K.m. is a h.K.m.

Solution of the fundamental conjecture (FC) for h.K.m.

Theorem FC (J. Dorfmeister, K. Nakajima, Acta Math. 1988) *A h.K.m. (M, g) is the total space of a holomorphic fiber bundle over a h.b.d. Ω. Moreover the fiber $\mathcal{F} = \mathcal{E} \times \mathcal{C}$ is (with the induced Kähler metric) the Kähler product of a flat homogeneous Kähler manifold $\mathcal{E}$ and a C-space $\mathcal{C}$.*

$$\mathcal{F} = \mathcal{E} \times \mathcal{C} \quad \overset{\text{Kähler}}{\hookrightarrow} \quad \begin{array}{c} M \\ \pi\downarrow \\ \Omega \end{array}$$

Remark. Ω is contractible so $M = \Omega \times \mathcal{F}$ as a complex manifold.

Main result on Kähler immersions into $\mathbb{C}^N$, $N \leq \infty$

Theorem 1.(Di Scala-Hishi-Loi) *Let* (M, g) *be a* n*-dimensonal h.K.m.. Then:*

(a) if (M, g) *can be Kähler immersed into* $\mathbb{C}^N$*,* $N < \infty$*, then* $(M, g) = \mathbb{C}^n$*;*

(b) if (M, g) *can be Kähler immersed into* $\ell^2(\mathbb{C})$*, then*

$$(M, g) = \mathbb{C}^k \times \mathbb{C}H^{n_1}_{\lambda_1} \times \cdots \times \mathbb{C}H^{n_l}_{\lambda_l},$$

where $k + n_1 + \cdots + n_l = n$*,* λ_j*,* $j = 1, \ldots, l$ *are positive real numbers and* $\mathbb{C}H^{n_j}_{\lambda_j} = (\mathbb{C}H^{n_j}, \lambda_j g_{hyp})$*(hence* $\mathbb{C}H^n_1 = \mathbb{C}H^n$*).*

Moreover, in case (a) (resp. (b)) the immersion is given, up to a unitary transformation of $\mathbb{C}^N$ (resp. $\ell^2(\mathbb{C})$), by the linear inclusion $\mathbb{C}^n \hookrightarrow \mathbb{C}^N$ (resp. by $(f_0, f_1, \ldots, f_l)$, where f_0 the linear inclusion $\mathbb{C}^k \hookrightarrow \ell^2(\mathbb{C})$ and each $f_k : \mathbb{C}H^{n_k} \to \ell^2(\mathbb{C})$ is λ_k times the map

$$z = (z_1, \ldots, z_{n_k}) \mapsto (\ldots, \sqrt{\frac{(j-1)!}{j!}} z_1^{j_1} \cdots z_{n_k}^{j_{n_k}}, \ldots), \qquad (1)$$

where $j = j_1 + \cdots + j_{n_k}$ and $j! = j_1! \cdots j_{n_k}!$.

Remark. Since a Kähler immersion is minimal, an alternative proof of (a) when $N < \infty$ follows by the work of A. J. Di Scala, Ann. Glob. Anal. Geom. 21 (2002). Assertion (b) is a generalization to arbitrary h.K.m. of the main theorem in A. J. Di Scala, A. Loi, Geom. Dedicata 125 (2007).

Main result on Kähler immersions into $\mathbb{C}H^N$, $N \leq \infty$

Theorem 2.(Di Scala-Hishi-Loi) *Let (M, g) be a n-dimensional h.K.m. If (M, g) can be Kähler immersed into $\mathbb{C}H^N$, $N \leq \infty$, then, up to a unitary transformation of $\mathbb{C}H^N$,*

$$(M, g) = \mathbb{C}H^n \hookrightarrow \mathbb{C}H^N.$$

Sketch of the proof. $(M, g) \to \mathbb{C}H^N$, $N \leq \infty \Rightarrow (M, g) \to \ell^2(\mathbb{C})$.

Theorem 1 $\Rightarrow$ $(M, g) = \mathbb{C}^k \times \mathbb{C}H^{n_1}_{\lambda_1} \times \cdots \times \mathbb{C}H^{n_l}_{\lambda_l}$.

By using the fact that $\mathbb{C}^k \not\to \mathbb{C}H^N, N \leq \infty$, the irreducibility of a Kähler immersion into $\mathbb{C}H^N$ and Calabi's rigidity theorem it follows that $(M, g) = \mathbb{C}H^n \hookrightarrow \mathbb{C}H^N$.$\Box$

Known results about immersions into $\mathbb{C}P^N$ with $N < \infty$

Theorem.(Takeuchi (Japan J. Math. 1978)) Let (M, g) be a h.K.m. which can be Kähler immersed into a finite dimensional complex projective space. Then M is compact, ω is integral ($[\omega]_{dR} \in H^2(M, \mathbb{Z})$), the immersion is injective and can be described in terms of the representation of semisimple Lie groups.

Remark. Viceversa if (M, g) is a compact Kähler manifold (not necessarily homogeneous) which can be Kähler immersed into a complex projective space $\mathbb{C}P^N$ one can assume $N < \infty$.

Moral. By Takeuchi's theorem and this remark, it remains to treat the case of noncompact h.K.m. which can be Kähler immersed into $\mathbb{C}P^\infty$.

First result on Kähler immersions into $\mathbb{C}P^\infty$

Theorem 3.(Di Scala-Hishi-Loi) *Let* (M, g) *be a h.K.m. which can be Kähler immersed into* $\mathbb{C}P^\infty$*. Then* ω *is integral,* M *is simply-connected and the immersion is injective.*

Sketch of the proof of Theorem 3

Let $f : (M, g) \to \mathbb{C}P^\infty$ be a Kähler immersion.

The integrality of $\omega = f^*\omega_{FS}$ is immediate since ω_{FS} is integral.

Theorem FC $\Rightarrow \mathcal{E} \hookrightarrow \mathcal{E} \times C = \mathcal{F} \hookrightarrow M \to \mathbb{C}P^\infty \overset{T=\mathbb{C}^n/\Lambda \not\to CP^\infty}{\Longrightarrow}$ $\mathcal{E} = \mathbb{C}^n \times C \Rightarrow M = \Omega \times \mathbb{C}^n \times C$ is simply-connected.

Calabi's rigidity theorem $\Rightarrow f \circ g = \mathcal{U}_g \circ f$, $\forall g \in G \Rightarrow f(M)$ is a h.K.m. $\Rightarrow f(M) \subset \mathbb{C}P^\infty$ is simply-connected.

$f : M \to f(M)$ is a local isometry $\Rightarrow f$ is a covering map $\Rightarrow f$ is injective.$\Box$

Second result on Kähler immersions into $\mathbb{C}P^\infty$

Theorem 4.(Di Scala-Hishi-Loi) *Let* (Ω, g_Ω) *be a h.b.d. Then there exists* $\lambda_0 \in \mathbb{R}^+$ *such that* $(\Omega, \lambda_0 g_\Omega)$ *can be Kähler immersed into* $\mathbb{C}P^\infty$*. Moreover, if* $(\Omega, \lambda g_\Omega)$ *can be Kähler immersed into* $\mathbb{C}P^\infty$ *for all* $\lambda > 0$*, then* $(\Omega, g_\Omega) = \mathbb{C}H^{n_1}_{\lambda_1} \times \cdots \times \mathbb{C}H^{n_l}_{\lambda_l}$.

Ingredients for the proof. Unitary representation of semisimple Lie groups; reproducing kernels of weighted Bergman spaces.

Sketch of the proof of Theorem 1 (based on Theorem 4)

Assume that (M, g) can be Kähler immersed into $\mathbb{C}^N$, $N \leq \infty$, we need to prove that: $(M, g) = \mathbb{C}^k \times \mathbb{C}H^{n_1}_{\lambda_1} \times \cdots \times \mathbb{C}H^{n_l}_{\lambda_l}$.

Calabi's rigidity + Riemannian geometry $\Rightarrow (M, g) = \mathbb{C}^k \times (\Omega, g_\Omega)$.

$\Rightarrow (\Omega, g_\Omega)$ can be Kähler immersed into $\mathbb{C}^N$, $N \leq \infty$.

it follows by a result of S. Bochner (Bull.Amer.Math.Soc., 1947) that, for all $\lambda > 0$, $(\Omega, \lambda g_\Omega)$ can be Kähler immersed into $\mathbb{C}P^\infty$.

Theorem 4 $\Rightarrow (\Omega, g_\Omega) = \mathbb{C}H^{n_1}_{\lambda_1} \times \cdots \times \mathbb{C}H^{n_l}_{\lambda_l} \Rightarrow (M, g) = \mathbb{C}^k \times \mathbb{C}H^{n_1}_{\lambda_1} \times \cdots \times \mathbb{C}H^{n_l}_{\lambda_l}$. $\square$

Integral forms and Kähler immersions into $\mathbb{C}P^\infty$

Question. If (M, g) is a h.K.m. such that ω is integral. Is it true that (M, g) can be Kähler immersed into $\mathbb{C}P^N$ for some $N \leq \infty$?

The Wallach set of a bounded symmetric domain

The Wallach set $W(\Omega) \subset \mathbb{R}$ of a bounded symmetric domain $\Omega \subset \mathbb{C}^n$ is a subset of $\mathbb{R}$ which "looks like ":

$$\underbrace{0 \quad . \quad . \quad . \quad .}_{\uparrow} \;\text{—————————————}$$

discrete part of $W(\Omega)$

Important property of the Wallach set: $W(\Omega) = \mathbb{R}$ (and hence the discrete part of $W(\Omega)$ is empty) iff and only if $\Omega = \mathbb{C}H^n$.

The Wallach set and immersions into $\mathbb{C}P^{\infty}$

Theorem W.(Loi-Zedda, Math.Ann., 2010) *Let* (Ω, g_B) *be a bounded symmetric domain. Then* $(\Omega, \lambda g_B)$ *can be Kähler immersed into* $\mathbb{C}P^{\infty}$ *if and only if* $\lambda\gamma \in W(\Omega) \setminus \{0\}$*, where* $\gamma > 0$ *denotes the genus of* Ω*.*

Three consequences of Theorem W

First consequence: (negative answer to the previous question) Let $(\Omega, g_B) \neq \mathbb{C}H^n$ be a bounded symmetric domain. Thus one can find $\lambda > 0$ such that $\lambda\gamma \notin W(\Omega)$:

$$0 \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad * \quad \text{———————————}$$

$$\uparrow$$

$$\lambda\gamma \notin W(\Omega)$$

By Theorem W, λg_B is not projectively induced (and $\lambda\omega_B$ is integral since Ω is contractible).

Second consequence: The complex hyperbolic space is the only bounded symmetric domains (Ω, g_B) where λg_B is projectively induced, for all $\lambda > 0$.

Third consequence: Let (Ω, g_B) be a bounded symmetric domain. Then, for $\lambda > 0$ suffciently large, λg_B is projectively induced.

Two conjectures

Conjecture 1: *Let (M, g) be a simply-connected h.K.m. such that its associated Kähler form ω is integral. Then there exists $\lambda_0 \in \mathbb{R}^+$ such that $\lambda_0 g$ is projectively induced.*

Remark. The integrality of ω in the conjecture is important since there exist simply-connected h.K.m. (M, g) such that $\lambda\omega$ is not integral for any $\lambda \in \mathbb{R}^+$ (and hence, a fortiori, λg is not projectively induced). Take, for example, $(M, g) = (\mathbb{C}P^1, g_{FS}) \times (\mathbb{C}P^1, \sqrt{2} g_{FS})$.

Conjecture 2: *Let (M, g) be a simply-connected h.K.m. such that its associated Kähler form ω is integral. If λg is projectively induced for all $\lambda \in \mathbb{R}^+$ then $(M, g) = \mathbb{C}^k \times \mathbb{C}H^{n_1}_{\lambda_1} \times \cdots \times \mathbb{C}H^{n_l}_{\lambda_l}$.*