LA DIASTASIS DI EUGENIO CALABI

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(V, ω) <u>varietà di Kähler reale analitica</u>

$$\omega_{|U} = \frac{i}{2}\partial\bar{\partial}\Phi, \ U \subset V$$

$\Phi : U \to \mathbb{R}$ *potenziale Kähleriano di* ω

$$\frac{i}{2}\partial\bar{\partial}\Phi = \frac{i}{2}\partial\bar{\partial}\Psi \Longrightarrow \Phi = \Psi + \varphi + \bar{\varphi}, \ \varphi \in \mathrm{Hol}(U)$$

Φ reale analitico $\Longrightarrow \exists \ \tilde{\Phi} : W \subset U \times U \to \mathbb{C}$

$$\tilde{\Phi}(p, \bar{q}) \qquad \Phi(p) = \tilde{\Phi}(p, \bar{p}) \qquad \tilde{\Phi}(p, \bar{q}) = \overline{\tilde{\Phi}(q, \bar{p})}$$

Diastasis (Calabi, Ann. of Math. 1953)

Supponiamo $W = U \times U$

$$D : U \times U \to \mathbb{R}$$

$$\boxed{D(p,q) = \tilde{\Phi}(p,\bar{p}) + \tilde{\Phi}(q,\bar{q}) - \tilde{\Phi}(p,\bar{q}) - \tilde{\Phi}(q,\bar{p})}$$

$$\Downarrow \quad \Downarrow \quad \Downarrow \quad \Downarrow \quad \Downarrow \quad \Downarrow \quad \Downarrow$$

$$D(p,q) = D(q,p) \qquad D(p,p) = 0 \qquad \frac{i}{2}\partial\bar{\partial}D(p,\cdot) = \omega$$

Proprietà fondamentale della diastasis

$$f : (V,\omega) \to (W,\Omega) \text{ olomorfa}$$

$$\boxed{f^*(\Omega) = \omega \Longleftrightarrow D_V(p,q) = D_W(f(p), f(q))}$$

DIASTASIS DEGLI SPAZI DI FORME

Esempio 1. $(\mathbb{C}^n, \omega = \frac{i}{2}\sum_{j=1}^n dz_j \wedge d\bar{z}_j)$

$$\omega = \frac{i}{2}\partial\bar{\partial}|z|^2 \Longrightarrow \ \Phi(z) = |z|^2 \Longrightarrow \tilde{\Phi}(z, \bar{w}) = z \cdot \bar{w}$$

$$D(z, w) = |z|^2 + |w|^2 - z \cdot \bar{w} - w \cdot \bar{z} = |z - w|^2$$

Esempio 2. $(\mathbb{C}P_b^n, \omega_{FS}), \ b > 0$

$$p_0 = [1, 0, \dots, 0] \in U_0 = \{Z_0 \neq 0\}$$

$$D(p_0, z) = \frac{1}{b}\log(1 + b|z|^2), \ z_j = \frac{Z_j}{Z_0}$$

Esempio 3. $(\mathbb{C}H_b^n, \omega_{hyp}), \ b < 0$

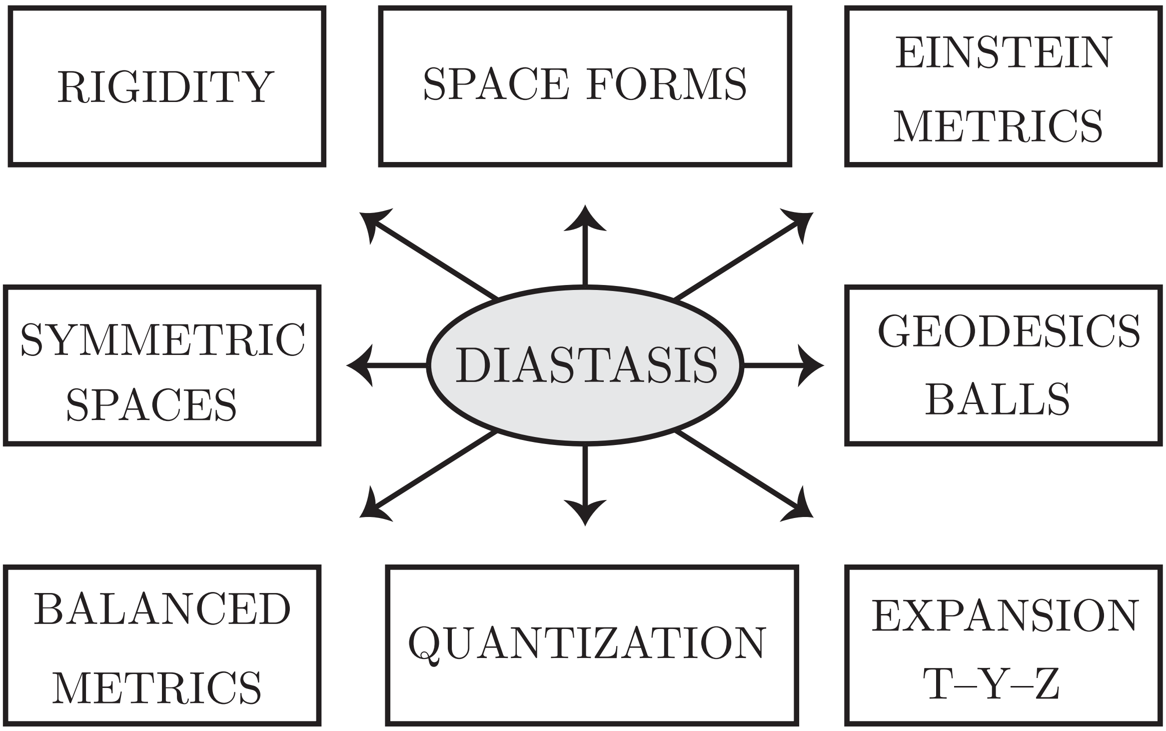
$$\mathbb{C}H_b^n = \{z \in \mathbb{C}^n | \ |z|^2 < -\frac{1}{b}\}$$

$$D(0, z) = \frac{1}{b}\log(1 + b|z|^2)$$

Esempio 4. Spazi di Fubini $F(n, b), n \leq \infty, b \in \mathbb{R}$
$F(n, 0) = \mathbb{C}^n, F(n, b) = \mathbb{C}P_b^n, \mathbb{C}H_b^n$.

'One could formulate several conjectures on the behaviour of the diastasis in the large, which in the author's opinion would furnish ample material for future study' (Calabi 1953)

'The diastasis has been rarely used up to now but it might still have a rich future' (Berger 2000)


RIGIDITY
SPACE FORMS
EINSTEIN METRICS
SYMMETRIC SPACES
DIASTASIS
GEODESICS BALLS
BALANCED METRICS
QUANTIZATION
EXPANSION T–Y–Z

RIGIDITA'

Teorema (Calabi, Ann. of Math. 1953)

$$f:(V,\omega)\to F(n,b),\quad g:(V,\omega)\to F(n',b)\ \ \underline{\text{full}}$$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$$n=n' \qquad\qquad\qquad g=U\circ f.$$

Teorema (M. Green, J. Diff. Geom. 1978)

$$f:(V,\omega)\to(W,\Omega),\quad g:(V,\omega)\to(W,\Omega)\ \ \underline{\text{non degeneri}}$$

$$\Downarrow$$

$$\exists\ \ U\in \mathrm{Aut}(W)\cap \mathrm{Symp}(W,\Omega)\ \ \text{tale che}\ \ g=U\circ f,$$

SPAZI DI FORME COMPLESSI

Teorema (Calabi, Ann. of Math. 1953)

$$\nexists\ \mathbb{C}H^k_b \to \mathbb{C}^n$$

$$\nexists\ \mathbb{C}^k \to \mathbb{C}P^n_b$$

$$\nexists\ \mathbb{C}H^k_b \to \mathbb{C}P^n_b$$

Teorema (Calabi, Ann. of Math. 1953)

$$F(n,b) \xrightarrow{f} F(n',b') \implies F(n,b) \overset{f}{\hookrightarrow} F(n',b')$$

Teorema (Umehara, Tokyo J. Math. 1987)

$$\nexists\ (V,\omega),\ \ (V,\omega) \hookrightarrow \mathbb{C}^n \quad \wedge \quad (V,\omega) \hookrightarrow \mathbb{C}H^n_b$$

$$\nexists\ (V,\omega),\ \ (V,\omega) \hookrightarrow \mathbb{C}^n \quad \wedge \quad (V,\omega) \hookrightarrow \mathbb{C}P^n_b$$

$$\nexists\ (V,\omega),\ \ (V,\omega) \hookrightarrow \mathbb{C}H^n_b \quad \wedge \quad (V,\omega) \hookrightarrow \mathbb{C}P^n_b$$

METRICHE DI EINSTEIN

Teorema *Le uniche sottovarietà compatte di Kähler--Einstein V^n di $\mathbb{C}P^{n+2}$ $(n \geq 2)$ sono:*

- $Q_n \subset \mathbb{C}P^{n+1} \subset \mathbb{C}P^{n+2}$ *(S. Chern, J. Differ. Geom. 1, 1967)*

- $\mathbb{C}P^n \subset \mathbb{C}P^{n+2}$ *(K. Tsukada, Math. Ann. 274, 1986)*

Teorema (D. Hulin, J. Geom. Anal. 10, 2000)

$$\text{KE compatta} \hookrightarrow \mathbb{C}P_b^n \implies c_1(KE) > 0$$

Teorema (M. Umehara, Tôhoku Math. J., 1987)

$$\text{KE} \xrightarrow{f} \mathbb{C}^n \text{ (oppure } KE \xrightarrow{f} \mathbb{C}H_b^n) \implies f \text{ tot. geod.}$$

DOMANDE SULLE METRICHE DI EINSTEIN I

Teorema (D. Hulin)

$$\text{KE compatta} \hookrightarrow \mathbb{C}P^n_b \implies c_1(KE) > 0$$

$$\Downarrow \quad \Downarrow \quad \Downarrow$$

D1 KE compatta $\hookrightarrow \mathbb{C}P^n_b \implies KE = S(simmetrico)$?

D2 KE $\rightarrow \mathbb{C}P^n_b$, scal $> 0 \implies KE \subset V$ compatta?

DOMANDE SULLE METRICHE DI EINSTEIN II

Ingrediente fondamentale nel teorema di Hulin:

Coordinate di Bochner (Bochner 1947)

$$\forall p \in V \ \exists \ (z_1, \dots, z_k) \text{ in } U \subset V, z_j(p) = 0$$

$$D(p,q) = |z(q)|^2 + \sum_{j,k} a_{j,k} z^j(q) \bar{z}^k(q), \quad a_{j,0} = a_{0,j} = 0.$$

$$D(p,q) = (\rho(p,q))^2 + O((\rho(p,q))^4)$$

D3 Quale è il legame tra le coordinate di Bochner di una varietà compatta e $c_1(KE)$?

Teorema (Arezzo-Loi, Sem. der Univ. Hamburg 74, 2004) Supponiamo che le le coordinate di Bochner $(z_1, \dots z_k)$ intorno ad un punto p di una varietà di KE compatta si estendano a funzioni olomorfe $(f_1, \dots f_k)$ su un aperto $U \subset KE$ tali che

$$\int_U \frac{i^k}{2^k} df_1 \wedge d\bar{f}_1 \wedge \cdots \wedge df_k \wedge d\bar{f}_k < \infty.$$

Allora $c_1(KE) > 0$.

SPAZI SIMMETRICI I

Teorema (H. Nakagawa and R. Takagi, J. Math. Soc. Japan 28, 1976)

$$\mathrm{LocS} \xrightarrow{f} \mathbb{C}^n \text{ (oppure } \mathrm{LocS} \xrightarrow{f} \mathbb{C}H_b^n) \implies f \text{ tot. geod.}$$

Teorema (M. Takeuchi, Japan J. Math 4, 1978)

$$\mathrm{LocS} \ \text{ completo} \xrightarrow{f} \mathbb{C}P_b^n \implies \mathrm{LocS} = S_+ \overset{f}{\hookrightarrow} \mathbb{C}P_b^n$$

Teorema (H. Tasaki, Osaka J. Math. 22, 1985)

$$\{q \in S_+ \mid D(p, \cdot) \text{ non è definita}\} = \mathrm{Cutlocus}(p)$$

Teorema (M. Engliš, preprint 2005) Sia (V, ω) una varietà di Kähler reale analitica.

$$\Delta_p(e^{-D(p,q)}) = \Delta_q(e^{-D(p,q)}) \iff (V, \omega) = \mathrm{LocS}$$

SPAZI SIMMETRICI II

Teorema (A. Loi, Diff. Geom. Appl. 2005) Sia (W, Ω) una varietà di Kähler *almost projective-like*, i.e. $e^{-D(x,\cdot)}$ è definita in W $\forall x \in W$. Allora

$$\text{LocS completo} \xrightarrow{f} (W, \Omega) \implies \text{LocS} = S \overset{f}{\hookrightarrow} (W, \Omega)$$

Corollario 1

$$\text{LocS completo} \xrightarrow{f} \hat{S} \implies \text{LocS} = S \overset{f}{\hookrightarrow} \hat{S}$$

Corollario 2

$$\text{LocS completo} \xrightarrow{f} F(N, b) \implies \text{LocS} = S \overset{f}{\hookrightarrow} F(N, b)$$

Teorema (A. Loi, Diff. Geom. Appl. 2005)

$$\nexists\ S_i \to S_j,\ i \neq j,\ i, j = -,\ 0,\ +$$

DOMANDE SUGLI SPAZI SIMMETRICI

Teorema (M. Umehara)

$$\mathrm{KE} \xrightarrow{f} \mathbb{C}^n \ \text{(oppure } \mathrm{KE} \xrightarrow{f} \mathbb{C}H^n_b) \implies \text{f tot. geod.}$$

$$\Downarrow \quad \Downarrow \quad \Downarrow$$

D1 $\mathrm{KE} \xrightarrow{f} S_- \implies$ f tot. geod.?

Teorema (Umehara)

$$\nexists\ (V,\omega),\ \ (V,\omega) \hookrightarrow \mathbb{C}^n \quad \wedge \quad (V,\omega) \hookrightarrow \mathbb{C}H^n_b$$

$$\nexists\ (V,\omega),\ \ (V,\omega) \hookrightarrow \mathbb{C}^n \quad \wedge \quad (V,\omega) \hookrightarrow \mathbb{C}P^n_b$$

$$\nexists\ (V,\omega),\ \ (V,\omega) \hookrightarrow \mathbb{C}H^n_b \quad \wedge \quad (V,\omega) \hookrightarrow \mathbb{C}P^n_b$$

$$\Downarrow \quad \Downarrow \quad \Downarrow$$

D2 $\exists\ (V,\omega),\ \ (V,\omega) \hookrightarrow \mathbb{C}^n \ \wedge\ (V,\omega) \hookrightarrow S_-$?

D3 $\exists\ (V,\omega),\ \ (V,\omega) \hookrightarrow \mathbb{C}^n \ \wedge\ (V,\omega) \hookrightarrow S_+$?

D4 $\exists\ (V,\omega),\ \ (V,\omega) \hookrightarrow S_- \ \wedge\ (V,\omega) \hookrightarrow S_+$?

GEODETICHE

Teorema (H. Tasaki)

$M_p = \{q \in S_+ \mid D(p, \cdot) \text{ non è definita}\} = \text{Cutlocus}(p)$

$$\Downarrow \quad \Downarrow \quad \Downarrow$$

Problema Studiare i legami tra il Cutlocus(p) di un punto $p \in (V, \omega)$ e M_p. Più in generale, studiare i legami tra la diastasis e l'applicazione esponenziale.

Teorema (A. Loi, Diff. Geom. Appl. Luglio 2005) Sia (V, ω, g) una varietà reale analitica. Esiste $U \subset TV$ e un embedding

$$\nu : U \to TV : \ (x, v) \mapsto (x, \nu_x(v))$$

$$\nu_x : T_xV \cap U \to T_xV \cap \nu(U)$$

tale che:

$$\boxed{D(x, \exp_x(\nu_x(v))) = g_x(v, v), \ (x, v) \in U}$$

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